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## Occurrence of the Efimov effect for three and more particles

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## Introduction

The Efimov effect was early discussed in nuclear physics

Nuclear halos were discovered in 1985 as spatially extended nuclei

Named in 1987 and related to weak binding of two-body systems

Three-body systems discussed in nuclei as smaller and less abundant

Halos were described by universal characteristics, scaling properties

Distinction between two and three-body halos, rms versus binding

Always ground or low-lying excited states, and finite range

For  $N > 3$  only exceptionally expected as halos

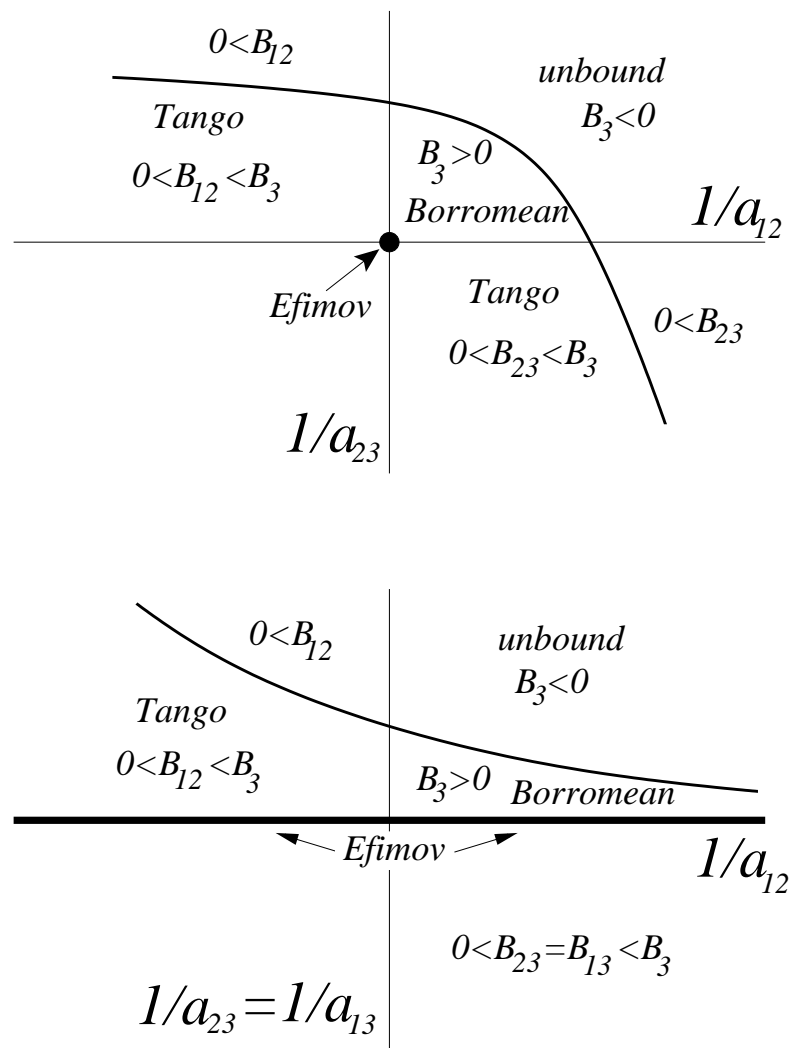


Figure 1: Different regions of stability for a three-body system. The  $s$ -wave two-body scattering lengths are  $a_{ik}$ . The central point,  $a_{ik} = \infty$  is the threshold for binding of the first state. All potentials are attractive or vanishing. The upper part assumes no interaction between particles 1 and 3 and the lower part assumes the same interaction between 1 – 3 and 2 – 3.

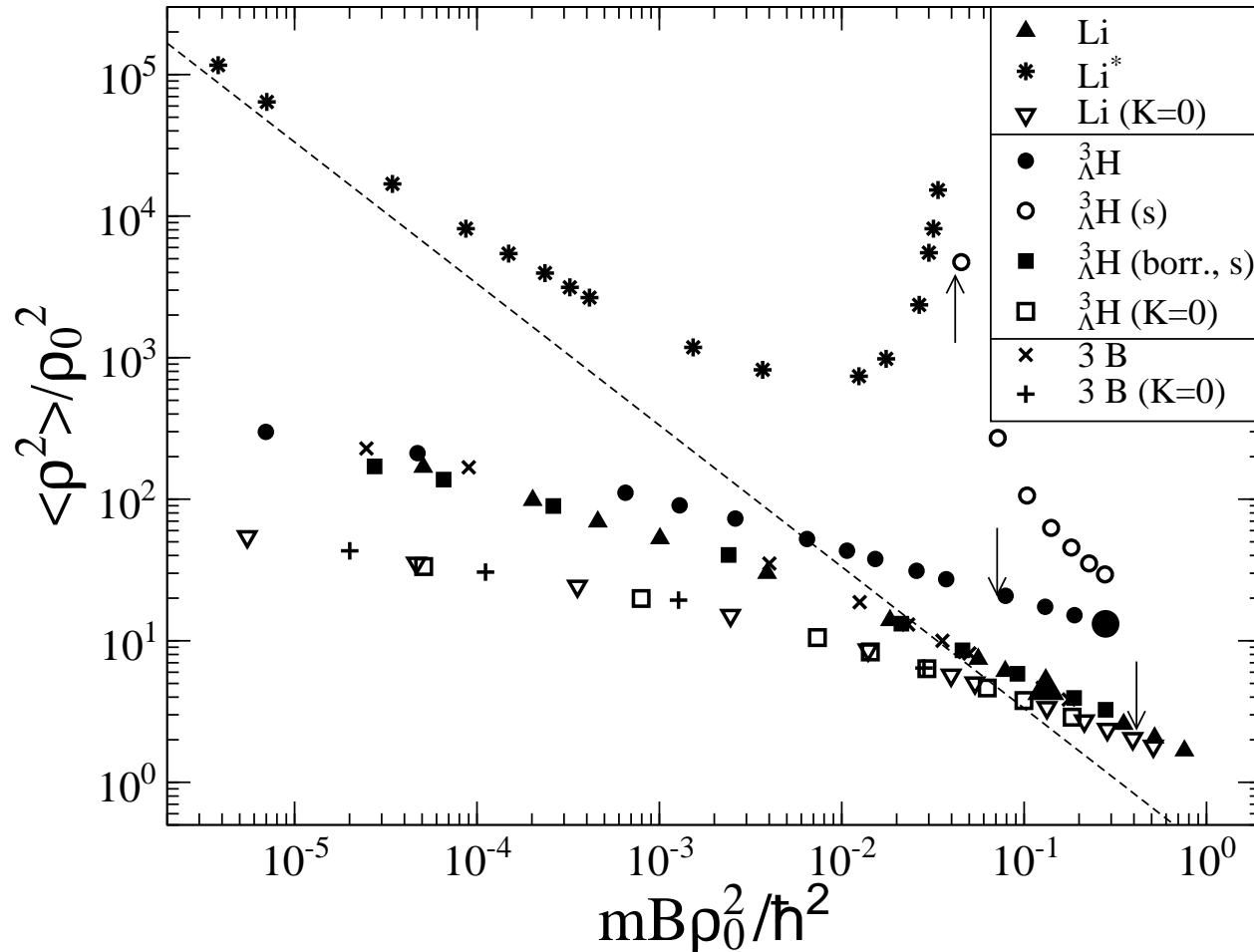


Figure 2: Scaling plot for three-body halos. The dashed line is the Efimov curve for  $\xi = 0$ ,  $\langle \rho^2 \rangle = (\xi^2 + 1)\hbar^2 / (3mB)$ . Triangles and stars are for masses of  ${}^{11}\text{Li}$  ( ${}^9\text{Li}+n+n$ ), squares and circles for  ${}^3_{\Lambda}\text{H}$  ( $\Lambda+n+p$ ). The realistic points have large closed triangle or square. Plus signs and crosses refer to three different particles with two fixed scattering lengths while the third is varied. The arrows indicate transitions between Borromean, tango and bound state regions. We used an average range for  $\rho_0$ .

## Hyperspherical method:

The Hamiltonian:

$$\hat{H} = \sum_{i=1}^3 \left( \frac{\hat{p}_i^2}{2m} + \frac{1}{2} m \omega^2 r_i^2 \right) + \sum_{i < j}^3 V(\vec{r}_{ij}) , \quad r_{ij} = |\vec{r}_j - \vec{r}_i|$$

$$\text{Hyper-radius: } \rho^2 = x^2 + y^2 = \frac{1}{m(m_i + m_k + m_j)} \sum_{i < j} m_i m_j r_{ij}^2$$

Hyper-radial equations for the eigenfunctions  $f_n$  and eigenvalues  $E_n$ :

$$\left( -\frac{\hbar^2}{2m} \frac{d^2}{d\rho^2} + U_n(\rho) - Q_{nn}^{(2)}(\rho) - E_n \right) f_n(\rho) = 0 ,$$

$$\frac{2mU_n(\rho)}{\hbar^2} \equiv -\frac{\xi_n^2 + 1/4}{\rho^2} ,$$

$$Q_{nn}^{(2)}(\rho) = \langle \Phi_n(\rho, \Omega) | \left( \frac{\partial}{\partial \rho} \right)^2 | \Phi_n(\rho, \Omega) \rangle_{\Omega} ,$$

Expectation value only includes angular integration for fixed  $\rho$

## Square well potential for $s$ -waves

Transcendental eigenvalue equation for large distance, identical bosons:

$$\begin{aligned} & \frac{\kappa}{i\xi} \cos(\alpha_0 \kappa) \left( \frac{8}{\sqrt{3}} \sin(i\pi\xi/6) \sin(i\alpha_0\xi) - i\xi \sin((\alpha_0 - \pi/2)i\xi) \right) \\ &= \sin(\alpha_0 \kappa) \left( \frac{8}{\sqrt{3}} \sin(i\pi\xi/6) \cos(i\alpha_0\xi) - i\xi \cos((\alpha_0 - \pi/2)i\xi) \right) \end{aligned}$$

$$\alpha_0 = \arcsin(R_0\sqrt{2}/\rho), \quad \kappa = \sqrt{-\xi^2 + \rho^2 2mV_0/\hbar^2}, \quad \frac{2mU(\rho)}{\hbar^2} \equiv -\frac{\xi^2 + 1/4}{\rho^2},$$

Expansion in  $1/\rho$ :

$$\frac{8}{\sqrt{3}} \sin(i\pi\xi/6) - i\xi \cos(\pi i\xi/2) = \frac{\rho\sqrt{2}}{a_s} \sin(\pi i\xi/2)$$

Efimov equation for vanishing right hand side

The **adiabatic potential for intermediate distances:**

$$U(\rho) = -\frac{\hbar^2}{2m} \left( \frac{\xi^2 + 1/4}{\rho^2} \right) \text{ for } R_e \leq \rho \leq a_{av} .$$

This potential has the generic form for the Efimov states

$\xi^2$  is a constant depending on the interactions.  $\frac{E_n}{E_{n+1}} = \frac{\langle \rho^2 \rangle_{n+1}}{\langle \rho^2 \rangle_n} = e^{2\pi n/\xi}$

$$a_{av}\sqrt{m} \equiv \frac{\sqrt{2}}{3} \sum_{i < k} \sqrt{\mu_{ik}} a_{ik}, \quad R_e\sqrt{m} \equiv \frac{\sqrt{2}}{3} \sum_{i < k} \sqrt{\mu_{ik}} R_{ik}, \quad \mu_{ik} = \frac{m_i m_k}{m_i + m_k}$$

$(a_{av}, R_e)$ : scattering length and effective range for identical particles

The **adiabatic potential for large distances:**

$$U(\rho) = \frac{\hbar^2}{2m\rho^2} \frac{48}{\pi\sqrt{2}} \frac{a_{av}}{\rho} \text{ for } a_{av} \leq \rho ,$$

**Small  $\rho$ :**  $U(\rho)$  is strongly interaction dependent

## Strength of the potential: $\xi$

For **identical bosons** the transcendental Efimov equation:

$$8 \sinh(\xi\pi/6) = \xi\sqrt{3} \cosh(\xi\pi/2)$$

The solution is  $\xi \simeq 1.00624$  independent of the particle masses

For **non-identical bosons** with **three large** scattering lengths:

$$\left( \frac{\xi \cosh(\xi\pi/2)}{2F} \right)^3 - \frac{\xi \cosh(\xi\pi/2)}{2F} \frac{(f_1^2 + f_2^2 + f_3^2)}{F^2} - 2 = 0$$

$$f_k = \frac{\sinh(\xi(\pi/2 - \varphi_k))}{\sin(2\varphi_k)}, \quad F = (f_1 f_2 f_3)^{1/3}, \quad \varphi_k = \arctan \left( \sqrt{\frac{m_k(m_1 + m_2 + m_3)}{m_i m_j}} \right)$$

For **non-identical bosons** with **two large** scattering lengths  
 $[a_{jk}$  and  $a_{ik}$  are large (not  $a_{ij}$ )]

$$\xi \cosh(\xi\pi/2) \sin(2\varphi_k) = 2 \sinh(\xi(\pi/2 - \varphi_k))$$

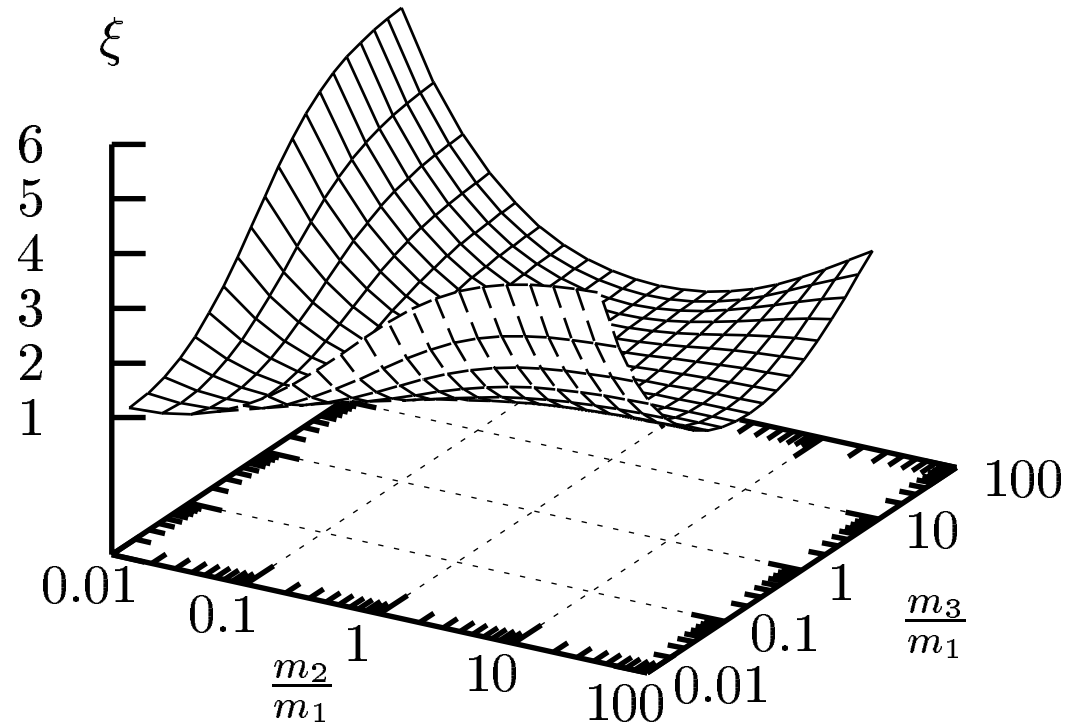


Figure 3: 3D plot of the strength parameter  $\xi$ :

$$U(\rho) = -\frac{\hbar^2}{2m} \left( \frac{\xi^2 + 1/4}{\rho^2} \right) \text{ as function of } m_2/m_1 \text{ and } m_3/m_1.$$

Large  $\xi$  (pronounced Efimov effect):

(i)  $m_1 \ll m_2, m_1 \ll m_3, m_2 \sim m_3$ , (ii)  $m_2 \ll m_1 \ll m_3$  and  $m_3 \ll m_1 \ll m_2$

Moderate  $\xi$  for symmetric systems

Exceedingly large  $\xi$  is possible for asymmetric systems

All three scattering lengths large, occur most likely for identical bosons

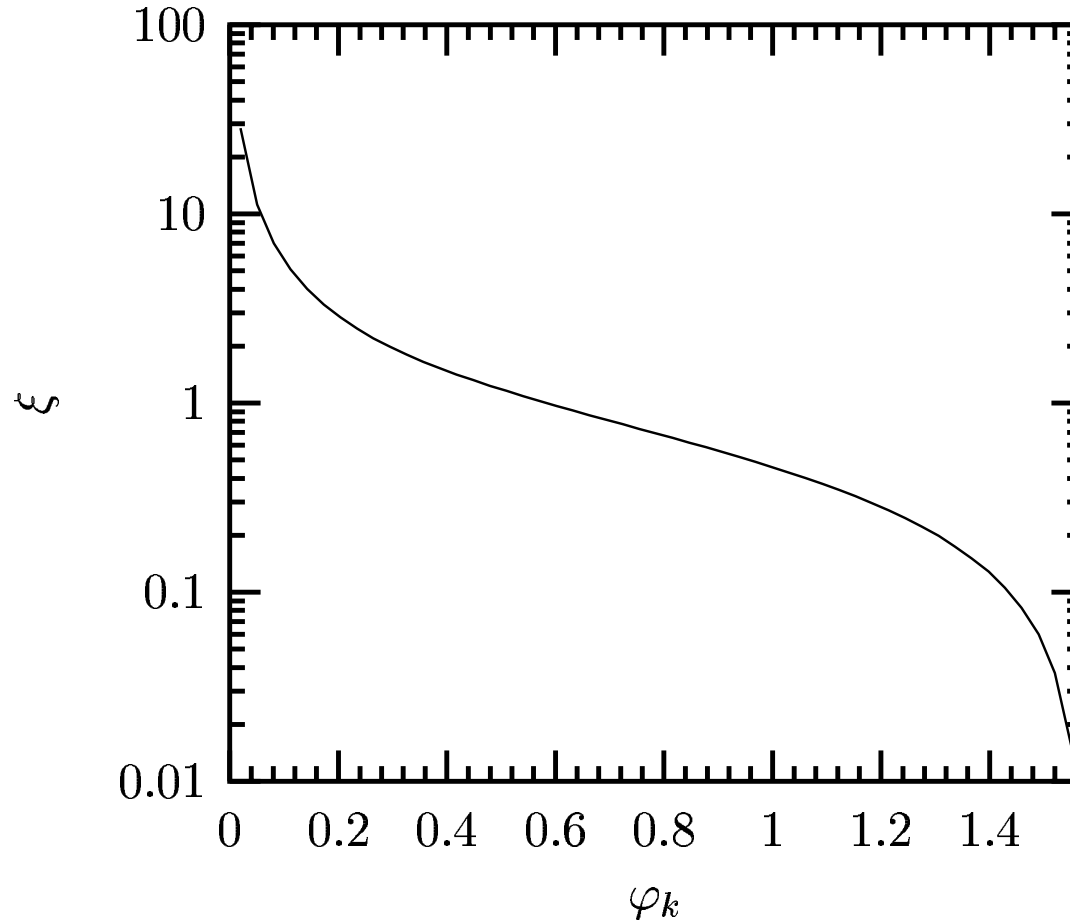


Figure 4: The strength parameter  $\xi$

$U(\rho) = -\frac{\hbar^2}{2m} \left( \frac{\xi^2 + 1/4}{\rho^2} \right)$  for two large scattering lengths as function of:

$$\varphi_k \equiv \arctan \left( \sqrt{\frac{m_k(m_1 + m_2 + m_3)}{m_i m_j}} \right)$$

Large  $\xi$  (pronounced Efimov effect)  $\varphi_k$  small:  $m_k \ll m_i$  and  $m_k \ll m_j$

Large  $a_{ik}$   $a_{jk}$  (not  $a_{ij}$ ) occur most likely when  $(i, j)$  are identical particles

## Examples

Most favorable case is one light and two heavy identical particles:

atom A + atom A + atom B, where  $m_B \ll m_A$

electron + atom A + atom B, where  $m_e \ll m_A$  and  $m_e \ll m_B$   
 (easily destroyed by mixing intrinsic (atom) degrees of freedom)

## Nuclei:

Coulomb destroys the effect leaving only two-neutron systems:

$^{11}\text{Li}$  ( $^9\text{Li} + n + n$ ),  $a_{nn} \approx 20$  fm, and  $\xi \approx 0.074$ , radial scaling factor  $3 \cdot 10^{18}$

Thus unlikely to occur naturally for bound states

## Excited states?

Two-body system:

${}^8\text{Be} (\alpha + \alpha)$  binding 0.0918 MeV

Three-body system:

${}^{12}\text{C} (\alpha + \alpha + \alpha)$  binding 0.38 MeV (Hoyle state)

In hot and dense astrophysical environment with electrons, i.e. plasma:

Coulomb screening  $\rightarrow$  reduced binding between  $\alpha - \alpha$

Closer to threshold AND Coulomb tail decreases

Efimov condition closer and triple  $\alpha$ -rate tremendously increased

## From finite range to zero-range correction for three particles

Assume identical bosons, only  $s$ -waves

Define:  $V_\rho(r) \equiv V\left(\frac{\rho}{\sqrt{\mu}} \sin\left(\frac{\sqrt{\mu}}{\rho} r\right)\right)$

$V$  is the two-body potential,  $\mu$  is the reduced mass

The phase shift  $\delta_\rho(k_\rho) \leftrightarrow V_\rho$ , where  $k_\rho = i\sqrt{\mu}\xi/\rho$

Generalized Efimov equation:

$$\frac{\sqrt{\mu}}{\rho} \cdot \frac{-i\xi \cos(i\xi \frac{\pi}{2}) + \frac{8}{\sqrt{3}} \sin(i\xi \frac{\pi}{6})}{\sin(i\xi \frac{\pi}{2})} = k_\rho \cot \delta_\rho(k_\rho) \quad \rightarrow \xi = \xi(\rho).$$

## Effective range expansion.

$V_\rho \rightarrow V(r)$  for  $\rho \rightarrow \infty$  and consequently  $\delta_\rho(k_\rho) \rightarrow \delta(k_\rho)$ .

Fix  $\rho$ , and use the low-energy effective range expansion of  $\delta_\rho(k_\rho)$

$$k_\rho \cot \delta_\rho(k_\rho) \approx -\frac{1}{a(\rho)} + \frac{R_e(\rho)}{2} k_\rho^2 \approx -\frac{1}{a} + R_E \frac{\mu}{\rho^2},$$

$$\frac{1}{a(\rho)} \approx \frac{1}{a} + \check{R} \frac{\mu}{\rho^2}, \quad R_e(\rho) = R_e + \check{R}_e \frac{\mu}{\rho^2}, \quad R_E = -\frac{R_e}{2} \xi^2 - \check{R},$$

For a finite-range potential  $V$

$$R_e = 2 \int_0^{r_0} (\tilde{u}(r)^2 - u(r)^2) dr, \quad \check{R} = \frac{1}{6} \int_0^{r_0} \frac{m}{\hbar^2} V'(r) r^3 u(r)^2 dr.$$

$u$  is the zero-energy solution to radial two-body Schrödinger equation, regular at the origin and normalized to match the free solution  $\tilde{u} = \sin(kr + \delta) / \sin \delta \rightarrow 1 - r/a$  at large  $r$  outside the potential.

## Efimov solution.

The Efimov equation,  $|a| \rightarrow \infty$ ,

$$i\xi \cos(i\xi \frac{\pi}{2}) = \frac{8}{\sqrt{3}} \sin(i\xi \frac{\pi}{6}) \text{ with the solution } \xi_0 = 1.00624.$$

Expansion of Efimov equation:

$$\frac{\sqrt{\mu}}{\rho} \cdot \frac{-i\xi \cos(i\xi \frac{\pi}{2}) + \frac{8}{\sqrt{3}} \sin(i\xi \frac{\pi}{6})}{\sin(i\xi \frac{\pi}{2})} = k_\rho \cot \delta_\rho(k_\rho) \rightarrow \xi = \xi(\rho).$$

to first order around  $\xi_0$  gives the potential:

$$V_{\text{eff}}^E(\rho) = \frac{\hbar^2 \mu}{m} \left[ \frac{-\xi_0^2 - 1/4 + 2R_E / (\xi_0^2 R_e)}{\rho^2} + \frac{2}{aR_e} - Q_{00}^E \right],$$

The coupling  $Q_{00}^E$  is necessary

Illustrate, Esry and Wang, picked a model potential

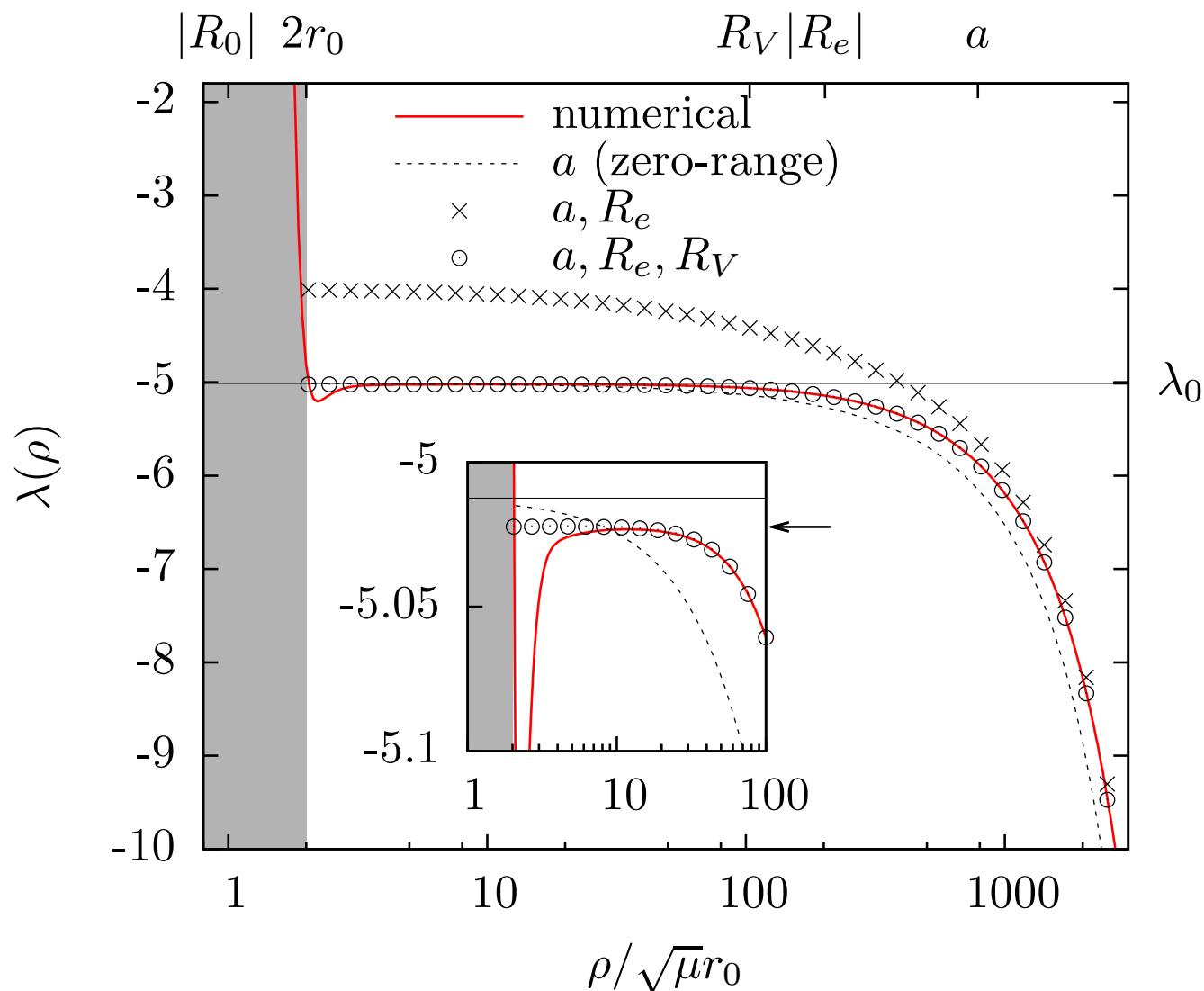


Figure 5: The adiabatic eigenvalues  $\lambda(\rho)$  for a potential with barrier, compared to solutions of the Efimov equation. The dotted line corresponds to the zero range model. The dashed line includes  $a, R_e$  and the long-dashed line includes  $a, R_e, \tilde{R}$ . The inset shows details near the Efimov solution,  $\lambda_0$ . The arrow indicates the correction  $\lambda_0 - 2R_E/R_e$ . The critical limit is  $\lambda = -4$ .

## Efimov conditions

The potential below the scattering length must be:

$$U(\rho) = -\frac{\hbar^2}{2m} \left( \frac{\xi^2 + 1/4}{\rho^2} \right), \text{ where } \xi > 0$$

We have:

$$V_{\text{eff}}^E(\rho) = \frac{\hbar^2 \mu}{m} \left[ \frac{-\xi_0^2 - 1/4 + 2R_E/(\xi_0^2 R_e)}{\rho^2} + \frac{2}{aR_e} - Q_{00}^E \right],$$

corresponding to  $\xi > 0$  in numerical illustration down **below**  $R_e$

**However**, the coupling  $Q_{00}^E$  receives contributions both from distances **inside** and **outside** the finite-range potential.

The condition is restored to be between  $R_e$  and  $a$

## **$N$ -body boson systems**

The Hamiltonian:

$$\hat{H} = \sum_{i=1}^N \left( \frac{\hat{p}_i^2}{2m} + \frac{1}{2} m \omega^2 r_i^2 \right) + \sum_{i < j}^N V(\vec{r}_{ij}) , \quad r_{ij} = |\vec{r}_j - \vec{r}_i|$$

Hyperspherical coordinates:

$$\rho^2 = \frac{1}{N} \sum_{i < j}^N r_{ij}^2 = \sum_i^N (\vec{r}_i - \vec{R})^2 = \sum_i^N r_i^2 - NR^2 , \quad \vec{R} = \sum_{i=1}^N \vec{r}_i / N$$

Radial equations for the eigenfunctions  $f$  and eigenvalues  $E$ :

$$\left( -\frac{\hbar^2}{2m} \frac{d^2}{d\rho^2} + U_\xi(\rho) - Q^{(2)}(\rho) - E \right) f(\rho) = 0$$

$$\frac{2mU_\xi(\rho)}{\hbar^2} = -\frac{\xi^2 + 1/4}{\rho^2} + \frac{(3N-4)(3N-6)-15}{4\rho^2} + \frac{\rho^2}{b_t^4} , \quad \text{trap length } b_t \equiv \sqrt{\hbar/(m\omega)}$$

$$Q^{(2)}(\rho) = \langle \Phi(\rho, \Omega) | \left( \frac{\partial}{\partial \rho} \right)^2 | \Phi(\rho, \Omega) \rangle_\Omega ,$$

## Zero-range interaction for $N$ bosons

$V$  is of zero range; Solutions are Jacobi functions  $\mathcal{P}_{i\xi}^{(a,b)}(x)$ ,

Boundary condition in terms of scattering length and effective range

$$\frac{\rho}{a_s} = \frac{-1}{\sqrt{2A}} [B + (\hat{R} - 1)\phi_{i\xi}(0)] = 0, \quad A, B \text{ and } R \text{ are analytical functions}$$

$$\rightarrow \text{eigenvalue or } \xi = \xi_N \text{ for each } N: \frac{2mU_\nu(\rho)}{\hbar^2} = -\frac{\xi_N^2 + 1/4}{\rho^2} + \frac{\rho^2}{b_t^4},$$

**N-body Efimov effect, Efimov condition:**

Asymptotic large  $N$  result:

$$\xi_N = \sqrt{\frac{5}{3}N^{7/3}\left(1 - \frac{2}{N}\right) - \frac{(3N-4)(3N-6)}{4} - \frac{1}{4}}$$

Scale factor:  $S_N = \exp(\pi/\xi_N)$  on radii

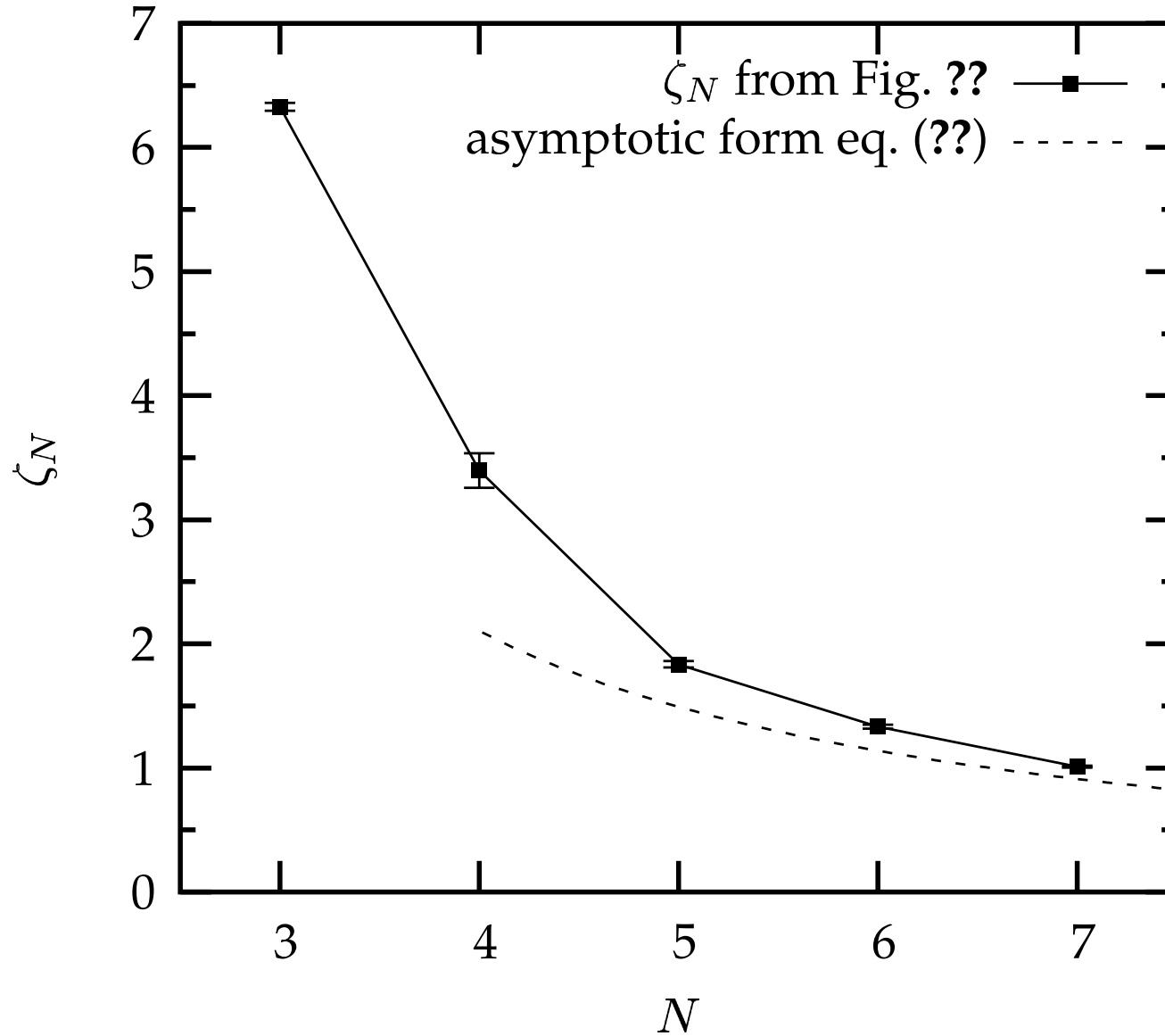


Figure 6: The exponent  $\zeta_N = 2\pi/\xi_N$  as function of the boson number  $N$ .

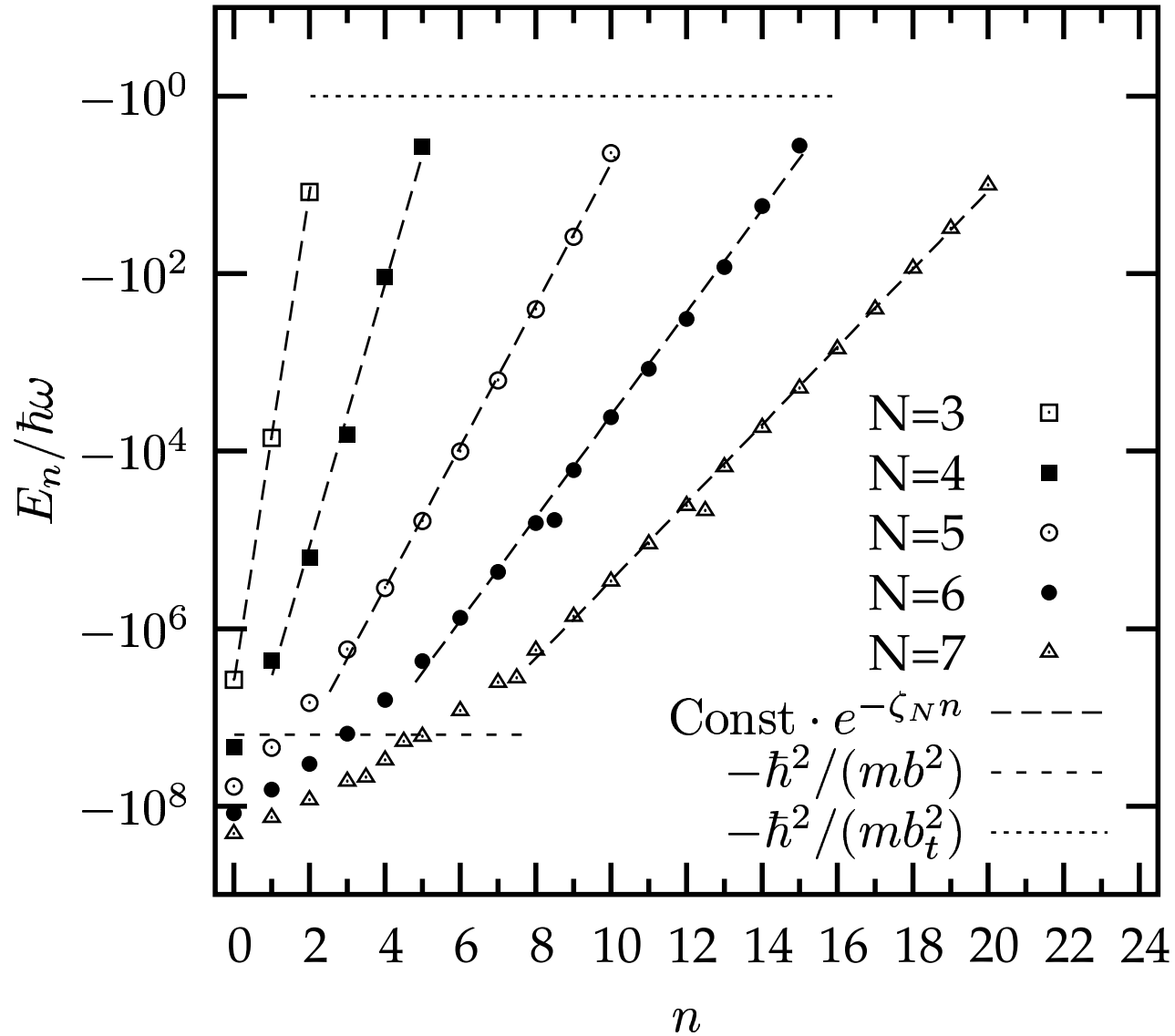


Figure 7: Energy  $E_n$  as function of the state number  $n$  for negative energy states for  $N$  bosons in a harmonic trap with length  $b_t$  interacting via an attractive potential of range  $b$  with scattering length much larger than  $b_t$ .

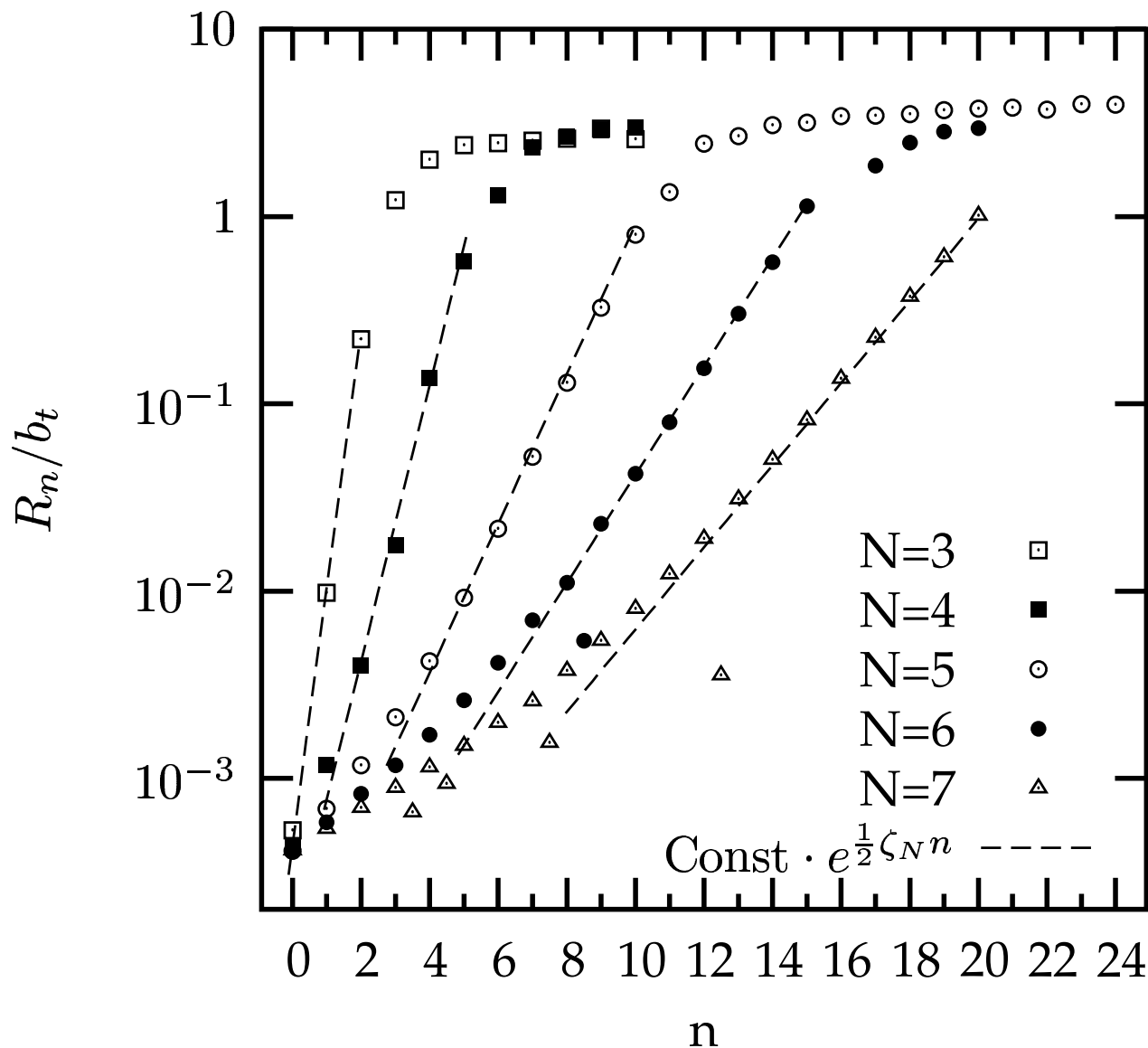


Figure 8: Root mean square radius as function of the state number  $n$  for negative energy states for  $N$  bosons in a harmonic trap with length  $b_t$  and two-body attraction of range  $b$  with scattering length much larger than  $b_t$ .

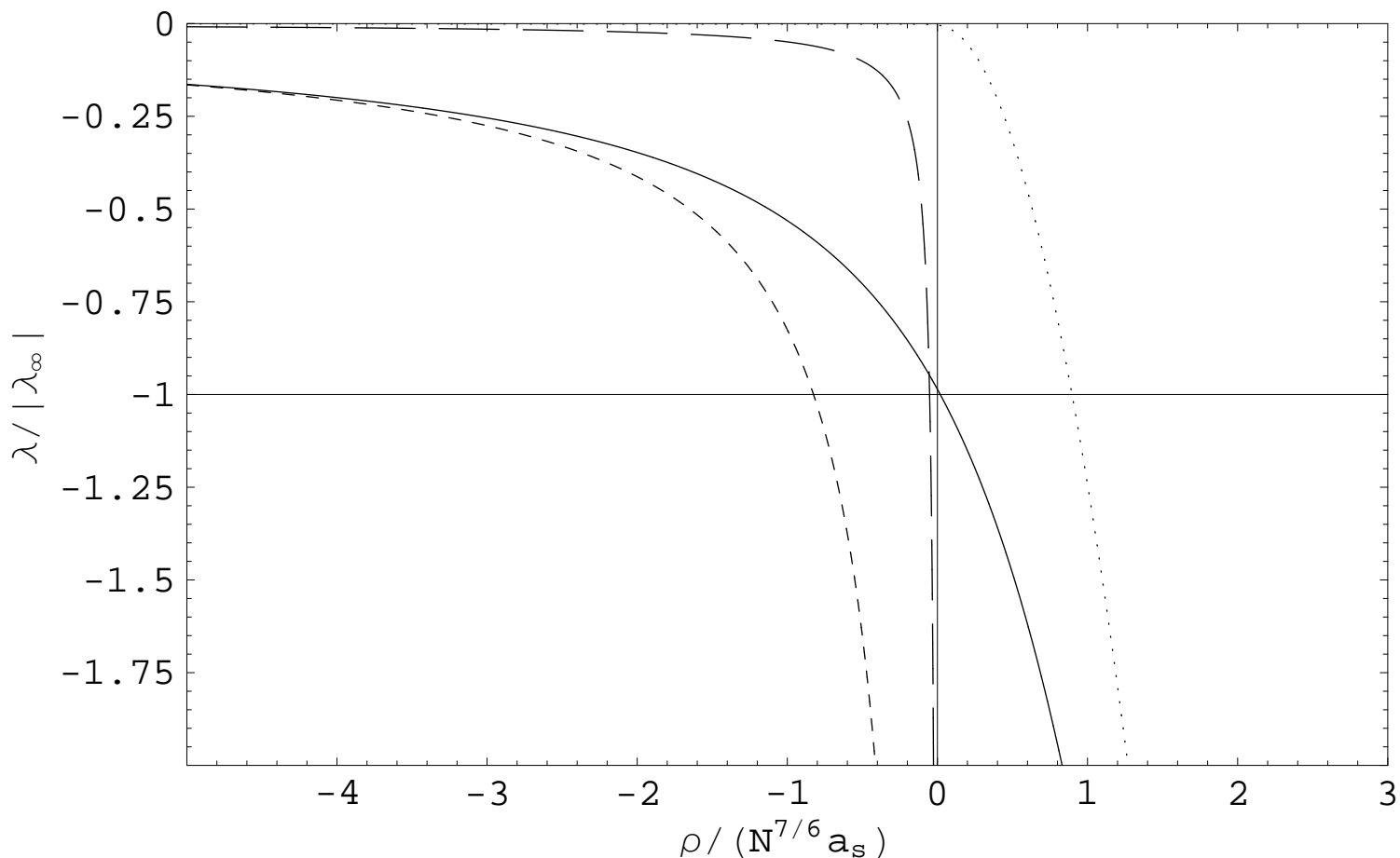


Figure 9: The angular eigenvalue,  $\lambda = -\xi^2 - (3N - 4)(3N - 6)/4 - 1/4$ , as function of  $\rho/a_s$ . The three different terms (12-term is dotted) (13-term is long-dashed) (34-term is short-dashed) are shown individually along with the sum (solid). The unit on the  $y$ -axis is defined by  $\lambda_\infty \approx -5/3N^{7/3}(1 - 2/N)$ , and the unit on the  $x$ -axis is scaled by  $N^{7/6}$ . Remarkably the sum of all contributions follows this scaling rule even though each of the three curves individually behaves differently.

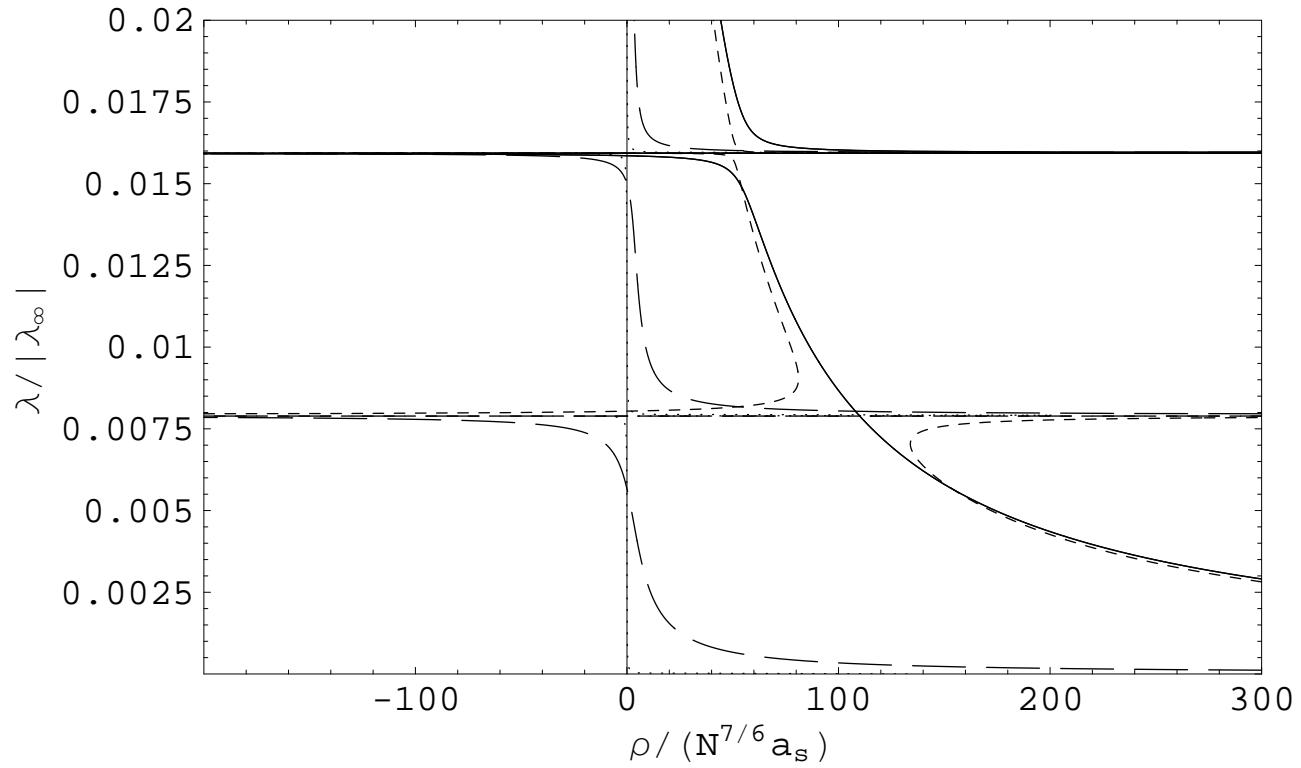


Figure 10: The positive angular eigenvalues as functions of  $\rho/a_s$ .

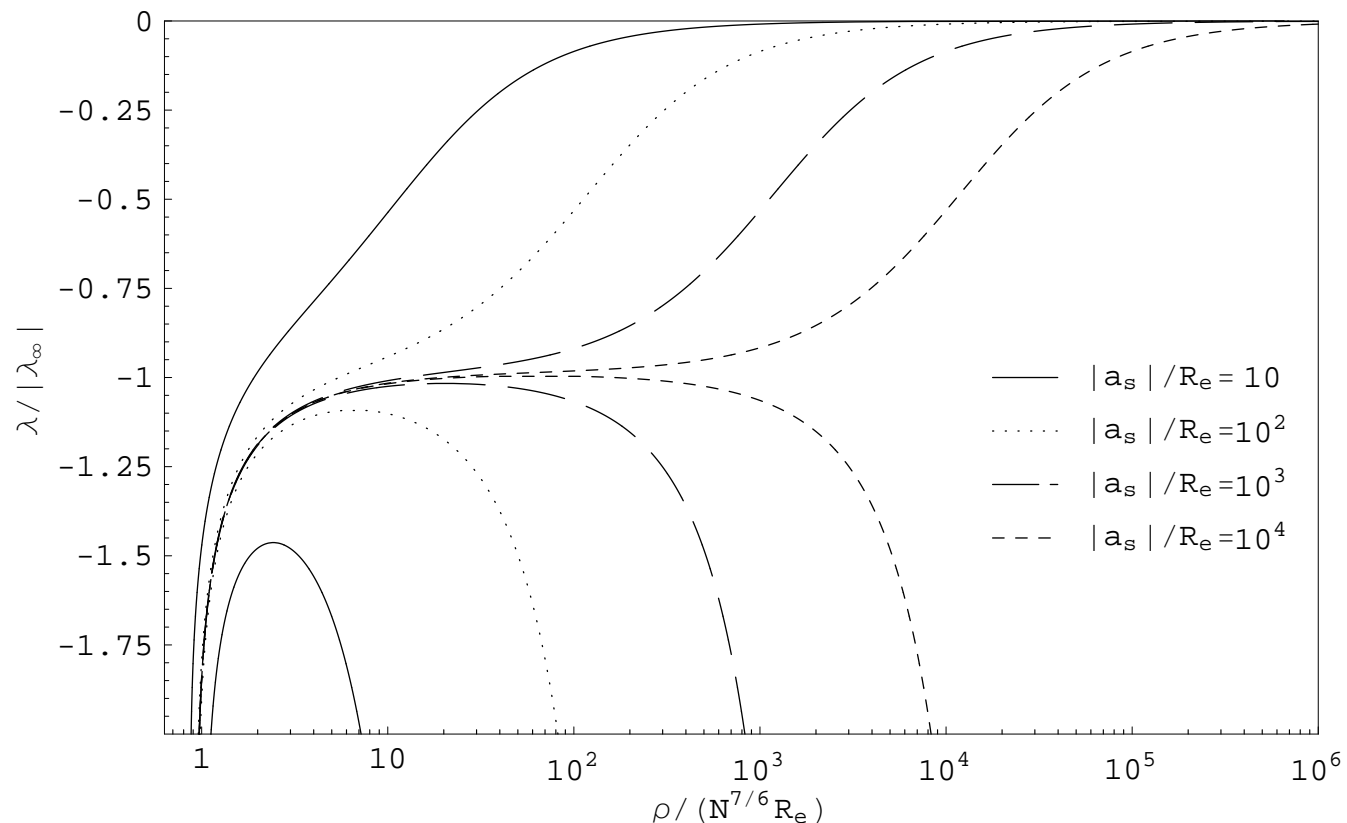


Figure 11: Negative angular eigenvalues for  $N = 100$  as functions of hyper-radius for several values of the scattering length  $a_s$  in units of the effective range  $R_e$ . The shape parameter  $P = 0$ . The unit of  $\lambda$  is as in fig. 3. The curves approaching zero or diverging for large  $\rho$  correspond to repulsive or attractive  $a_s$ , respectively.

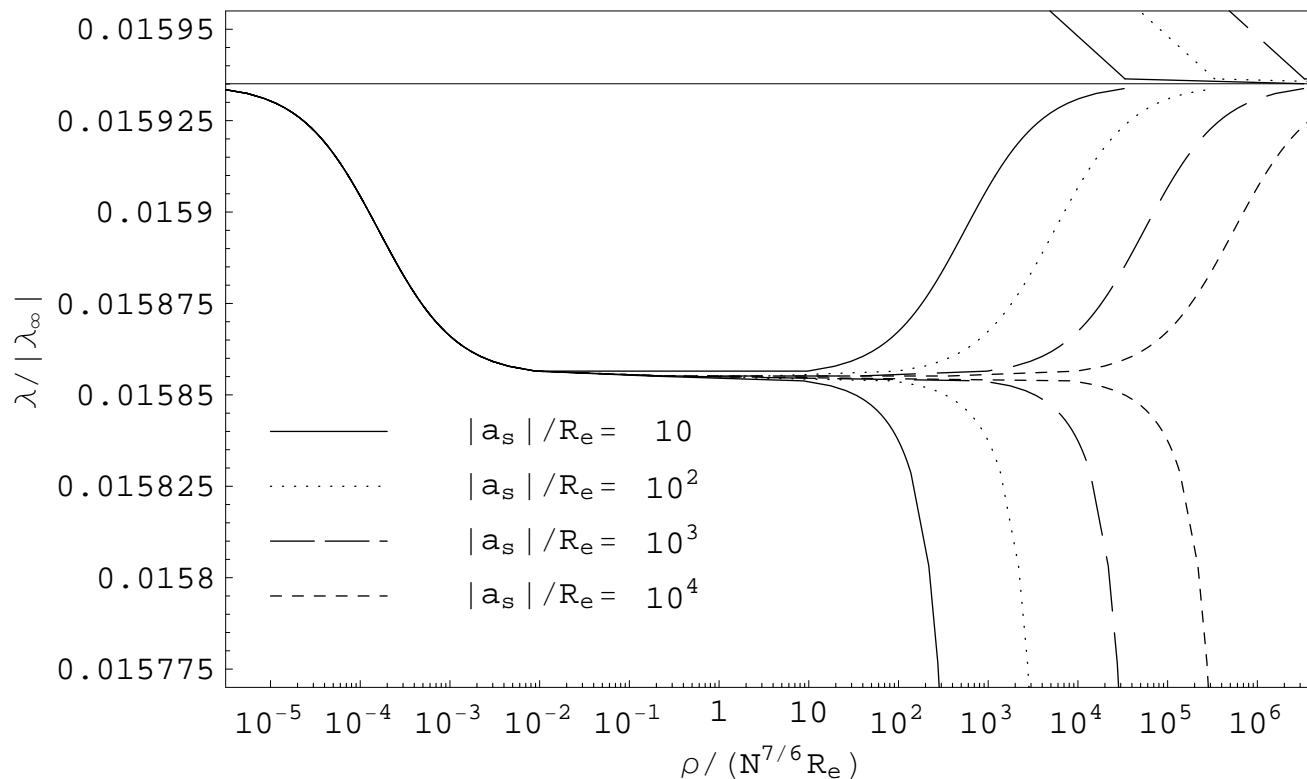


Figure 12: Positive angular eigenvalues for  $N = 100$  as functions of hyperradius for several values of the scattering length  $a_s$ , both in units of the effective range  $R_e$ . The shape parameter  $P = 0$ . The unit of  $\lambda$  is as in fig. 3. The horizontal  $\nu = 2$  is also shown.

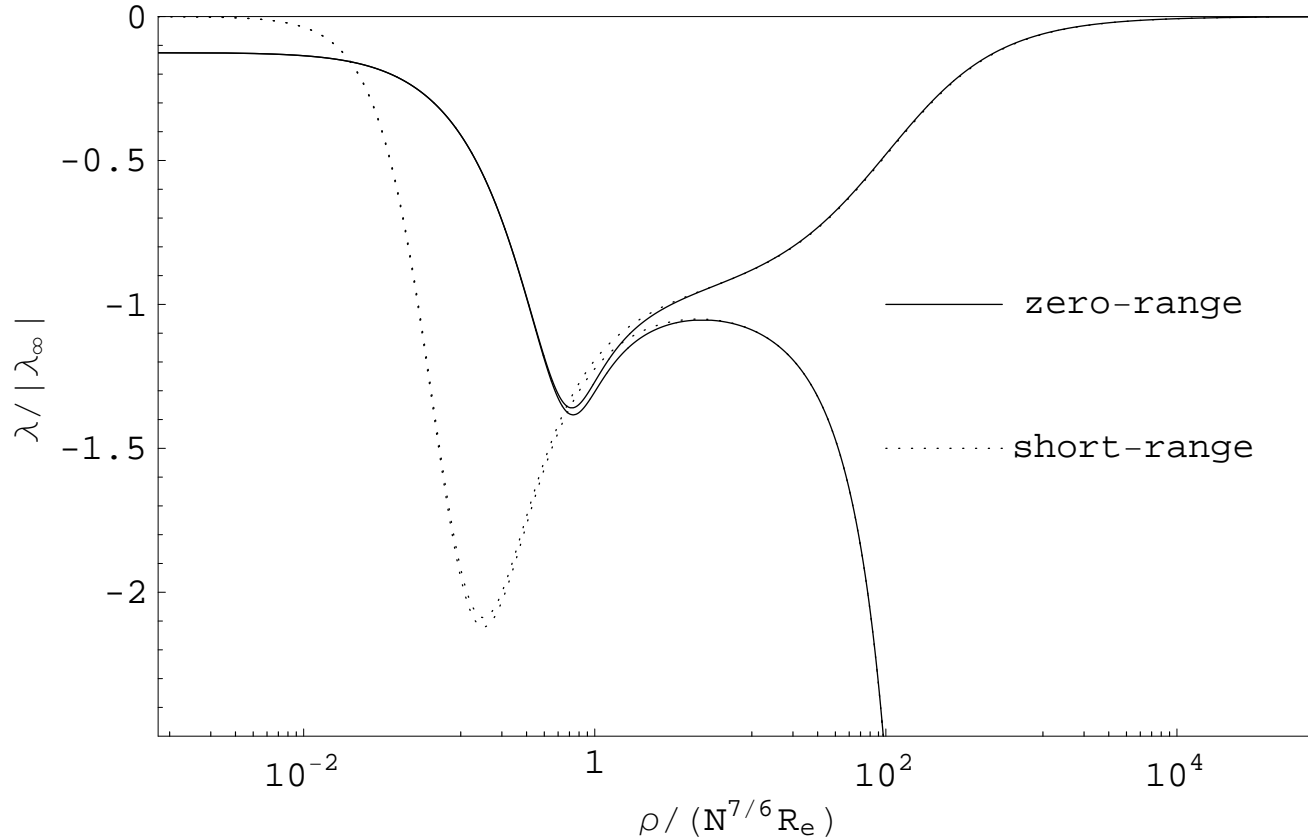


Figure 13: The angular eigenvalue for  $N = 10$  in the same units as in fig. ?? for zero-range and finite-range Gaussian two-body potentials with the same scattering lengths and effective ranges. The Gaussian is  $V_0 \exp(-2\rho^2 \sin^2 \alpha/b^2)$  with  $V_0 = 4\hbar^2 a_B/(mb^3 \sqrt{\pi})$ . For zero range we use  $a_s/R_e = \pm 100$  and  $P = 0.1$ , and for finite range we use  $a_B/b = -1.180, -1.199, R_e/b = 1.44, 1.43$ , where the pairs are for negative and positive scattering lengths respectively. The two curves of each type correspond to the different signs of the scattering lengths.

## Summary and Conclusions

1. Efimov scaling and Borromean scaling are different
2. Square well for three particles and  $s$ -waves is analytic
3. Efimov effect is “large” for large mass differences
4. Large mass differences is better than contribution from three large  $a$
5. Range corrections to zero-range limit have three ingredients
  - (i) second order correction to scattering length
  - (ii) effective range
  - (iii) non-adiabatic term
6.  $N$ -body zero-range radial potentials scale
7.  $N$ -body Efimov states appear with two-body correlations

# References

- [1] D. V. Fedorov and A. S. Jensen,  
Efimov effect in coordinate space Faddeev equations,  
*Phys. Rev. Lett.* **71**, 4103 (1993).
- [2] D.V. Fedorov, A.S. Jensen and K. Riisager,  
Efimov states in halo nuclei,  
*Phys. Rev. Lett.* **73**, 2817 (1994).
- [3] D.V. Fedorov, A.S. Jensen and K. Riisager,  
Three-body halos. I. Gross properties,  
*Phys. Rev.* **C49**, 201 (1994);  
II. From two- to three-body asymptotics,  
*Phys. Rev.* **C50**, 2372 (1994).
- [4] A.S. Jensen, D.V. Fedorov, K. Langanke and H.-M. Müller,  
Three alpha-particle system in a dense helium plasma,  
ENAM 95, Int. Conf. on exotic nuclei and atomic masses, eds. Simon and Sorlin, Edition  
Frontieres, 677 (1995).
- [5] E. Nielsen, D.V. Fedorov and A.S. Jensen,  
Three-body halos in two dimensions,  
*Phys. Rev.* **A56**, 3287-3290 (1997).

- [6] A.S. Jensen, E. Garrido and D.V. Fedorov,  
Three-Body Systems with Square-Well Potentials in  $L=0$  States, *Few-Body Systems*, **22**,  
193-236 (1997).
- [7] E. Nielsen, D.V. Fedorov and A.S. Jensen,  
The structure of atomic helium trimers: Halos and Efimov states,  
*J. Phys.* **B31**, 4085-4105 (1998).
- [8] E. Nielsen, D.V. Fedorov and A.S. Jensen,  
The Efimov effect and external fields,  
*Phys. Rev. Lett.* **82**, 2844-2847 (1999).
- [9] K. Riisager, D.V. Fedorov and A.S. Jensen,  
Quantum halos,  
*Europhysics Lett.* **49**, 547-553 (2000).
- [10] D.V. Fedorov and A.S. Jensen,  
Regularization of a three-body problem with zero range potentials,  
*J. Phys.* **A34**, 6003-6012 (2001).
- [11] E. Nielsen, D.V. Fedorov, A.S. Jensen and E. Garrido,  
The three-body problem with short-range interactions,  
*Phys. Rep.* **347** (2001) 373-459.
- [12] D.V. Fedorov and A.S. Jensen,  
Regularized zero-range model and an application to the triton and the hypertriton,  
*Nucl. Phys. A* **697**, 783-801 (2002).

- [13] A.S. Jensen, K. Riisager, D.V. Fedorov and E. Garrido,  
Classification of three-body quantum halos  
*Europhysics Lett.* **61**, 320-326 (2003).
- [14] D.V. Fedorov, E. Garrido, and A.S. Jensen,  
Complex scaling of the hyper-spheric coordinates and Faddeev equations,  
*Few-body systems*, **33** (2003) 153-171.
- [15] A.S. Jensen and D.V. Fedorov,  
Efimov states in asymmetric systems  
*Europhysics Lett.* **62** (2003) 336-342.
- [16] A.S. Jensen, K. Riisager, D.V. Fedorov and E. Garrido,  
Structure and reactions of quantum halos,  
*Rev. Mod. Phys.* **76** (2004) 215-261.
- [17] T. Sogo, O. Sørensen, A.S. Jensen and D.V. Fedorov  
Semi-analytic Faddeev solution to the N-boson problem with zero-range interactions  
*Europhys.Lett.* **69**, 732 (2005).
- [18] T. Sogo, A.S. Jensen and D.V. Fedorov  
Zero-range approximation for two-component boson systems  
*Few-Body Systems* **37**, 155-178 (2005).
- [19] E. Garrido, D.V. Fedorov and A.S. Jensen,  
Efimov effect in nuclear three-body resonance decays,  
*Phys. Rev. Lett.* **96** (2006) 112501(4).

- [20] E. Garrido, D.V. Fedorov and A.S. Jensen  
Efimov effect in nuclear three-body resonance decays  
*Phys. Rev. Lett.* **96**, 112501 (2006).
- [21] D.V. Fedorov and A.S. Jensen  
On the uniqueness of the solutions to the three-body problem with zero-range interactions  
*Few-body systems*, **38**, 75-78 (2006).
- [22] M. Thøgersen, D.V.Fedorov, and A.S. Jensen  
Trapped Bose gases with large positive scattering length,  
*Eur. Phys. Lett.* **79** (2007) 40002.
- [23] M. Thøgersen, D.V.Fedorov, and A.S. Jensen  
*N*-body Efimov states in trapped bosons *Europhys.Lett.* **83** (2008) 30012 (6pp).
- [24] M. Thøgersen, D.V.Fedorov, and A.S. Jensen  
Universal properties of Efimov physics beyond the scattering length approximation  
*Phys.Rev. A* **78** (2008) R020501.
- [25] M. Thøgersen, D.V.Fedorov, A.S. Jensen, B.R.Esry and Y. Wang  
Conditions for Efimov Physics for finite range potentials  
*Phys.Rev. A* **80** (2009) 013608, 1-4.