High Fidelity Quantum Computing via Dynamical Decoupling

ITAMP, Aug.11, 2010

Menu:

1. Intro to DD, CDD, QDD [PRL **95**, 180501 (2005), PRA **75**, 062310 (2007), PRL **104**, 130501 (2010)]

2. Decouple-while-compute [arXiv:0911.2398]

3. Decouple-then-compute [arXiv:0911.3202]

Joint work with:
Kaveh Khodjasteh (1), Jacob West (1,2), Bryan Fong (1,2), Mark Gyure (2)
Hui-Khoon Ng (3), John Preskill (3)

Dynamical Decoupling Basics

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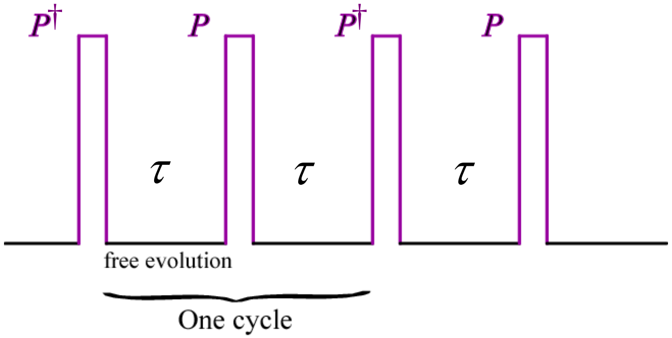
A pulse producing a unitary evolution P, such that

$$PH_{\mathrm{SB}}P^{\dagger} = -H_{\mathrm{SB}} \qquad \text{i.e., } \{P, H_{\mathrm{SB}}\} = 0$$

(Carr-Purcell, Hahn spin-echo)



Ideal (zero-width) pulses, and ignoring H_{B}:

$$P\exp(-i\tau H_{\mathrm{SB}})P^{\dagger}\exp(-i\tau H_{\mathrm{SB}}) = \exp(-i\tau PH_{\mathrm{SB}}P^{\dagger})\exp(-i\tau H_{\mathrm{SB}})$$
$$= \exp(i\tau H_{\mathrm{SB}})\exp(-i\tau H_{\mathrm{SB}}) = I$$

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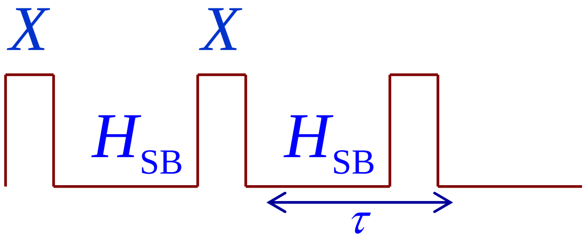

$$H_{\mathrm{SB}} = \lambda Z \otimes B$$

$XZX = -Z \quad \Rightarrow$
"time reversal",
H_{SB} averaged to zero
(in 1st order Magnus expan.)

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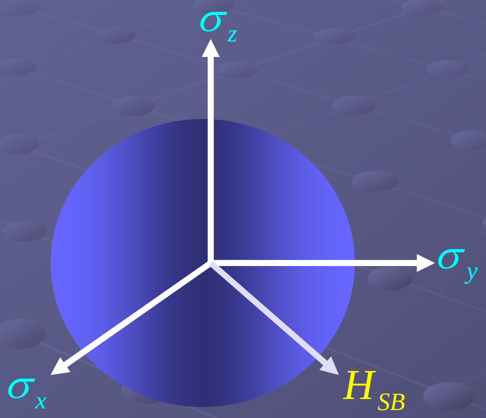
Dynamical Decoupling = Symmetrization

Viola & Lloyd Phys. Rev. A **58**, 2733 (1998); Byrd & Lidar, Q. Inf. Proc. **1**, 19 (2002)

System-bath Hamiltonian: $H_{SB} = \sum_{\alpha} \underset{\text{system}}{S_\alpha} \otimes \underset{\text{bath}}{B_\alpha}$



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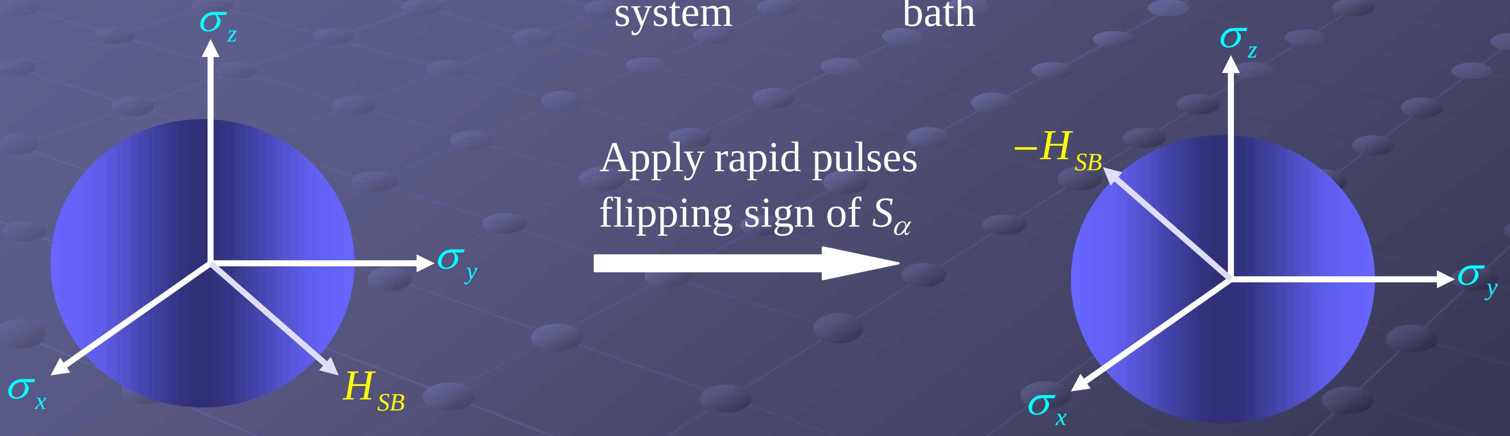

Apply rapid pulses flipping sign of S_α

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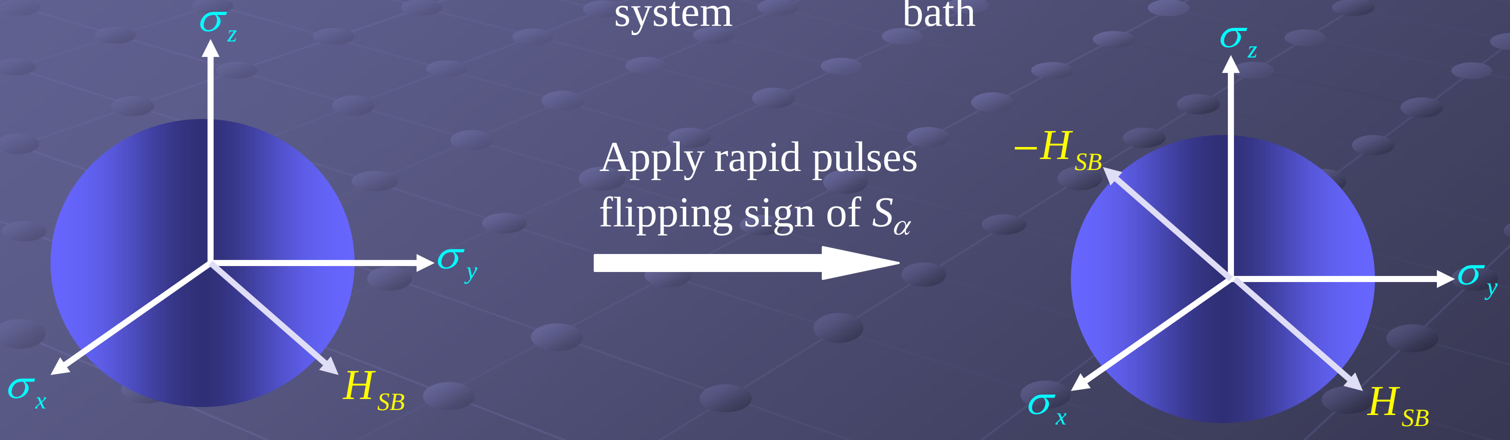

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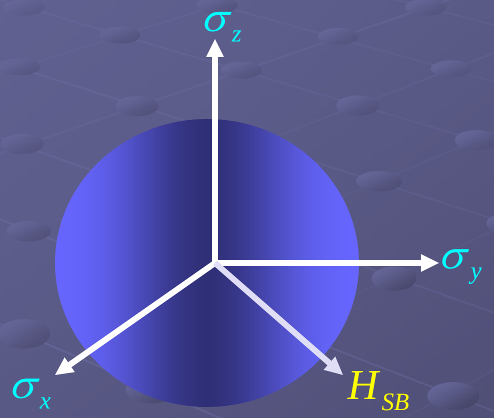

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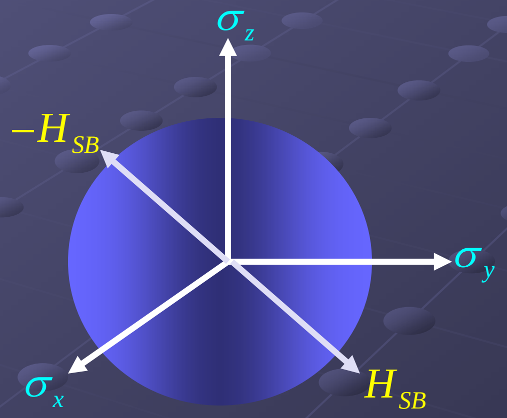

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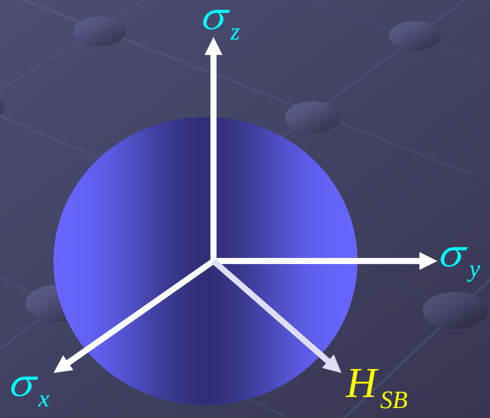

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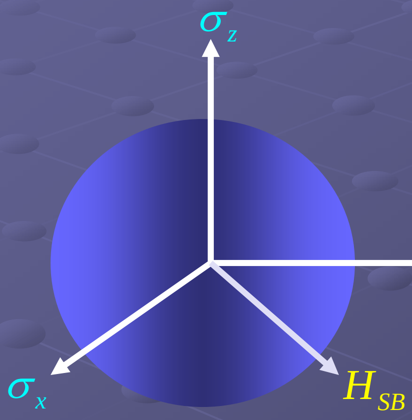

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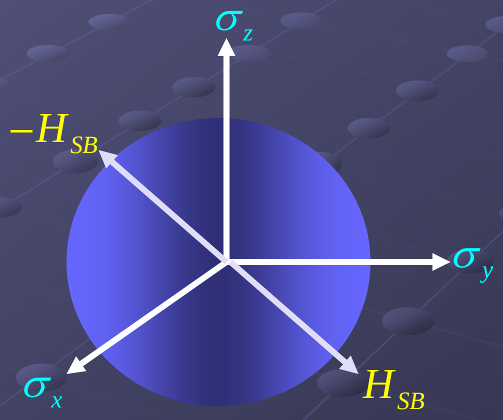

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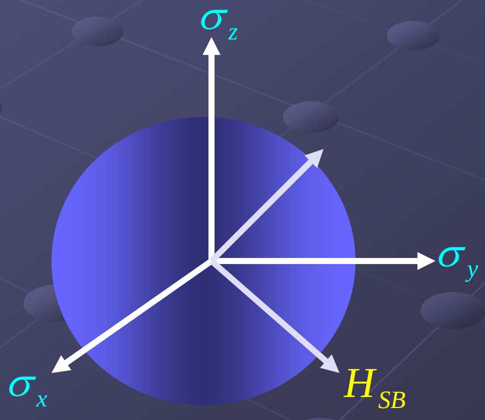

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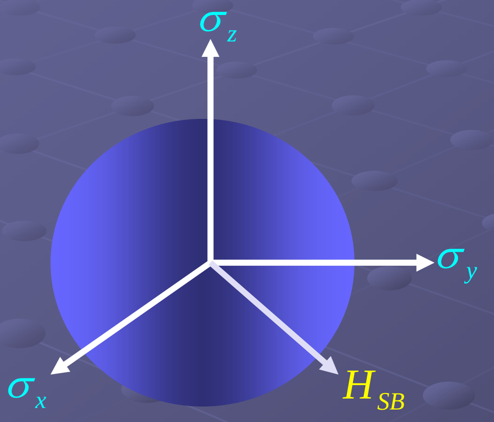

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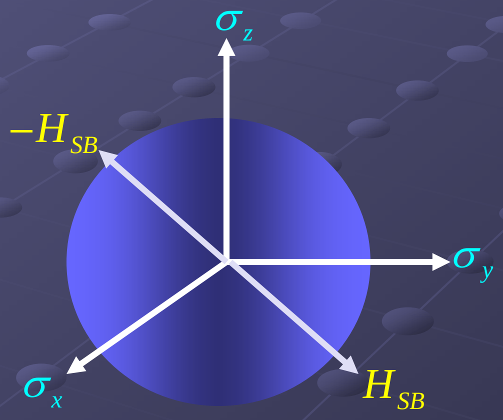

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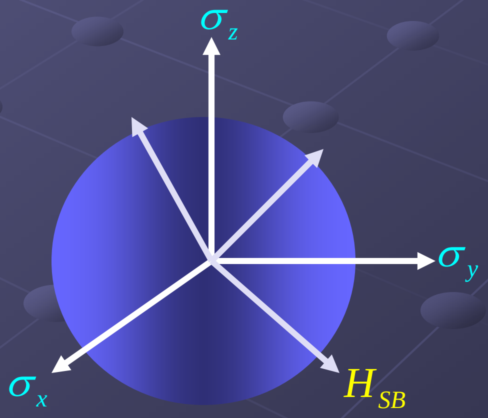

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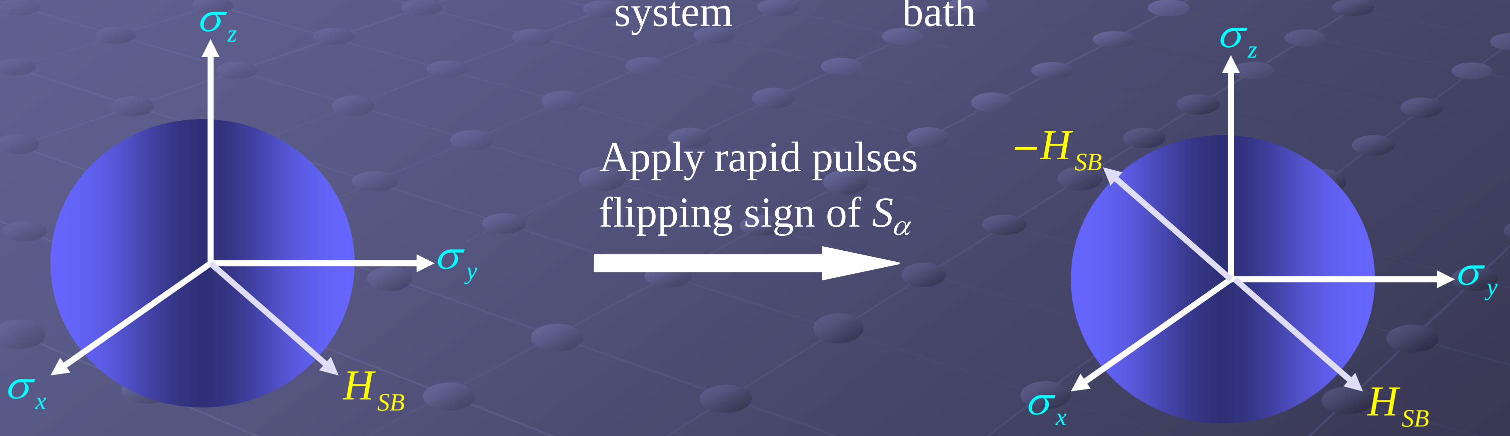

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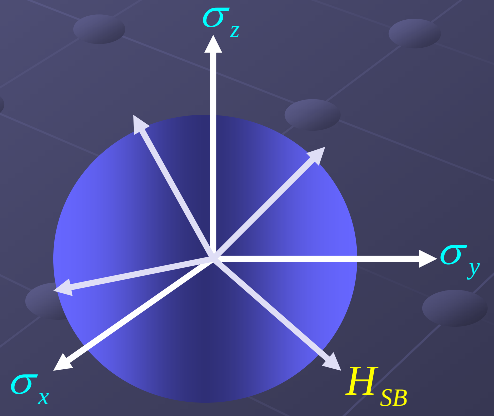

H_{SB} averaged to zero.

Dynamical Decoupling Theory

"Symmetrizing group" of pulses $\{ g_i\}$ and their inverses are applied in series:

$$\boxed{(g_N^\dagger \mathbf{f} g_N)\cdots(g_2^\dagger \mathbf{f} g_2)(g_1^\dagger \mathbf{f} g_1) \approx \exp(-i\tau \sum_i g_i^\dagger H_{SB} g_i)}$$

$$\mathbf{f} \equiv \exp(-iH_{SB}\tau)$$

first order Magnus expansion

Periodic DD: periodic repetition of the universal DD pulse sequence

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Choose the pulses so that:

$$H_{SB} \mapsto H_{\text{eff}}^{(1)} \equiv \sum_i g_i^\dagger H_{SB} g_i = 0$$

Dynamical Decoupling Condition

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Dynamical Decoupling Condition

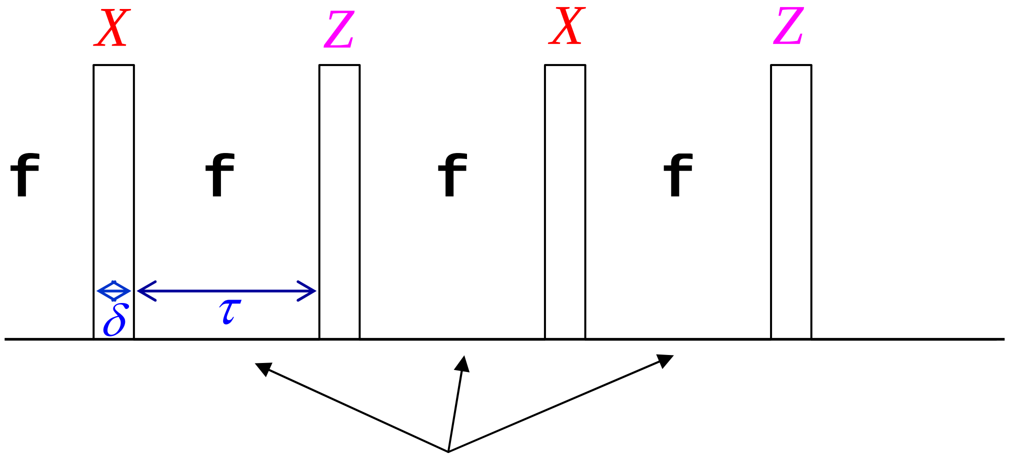
For a qubit the Pauli group $G=\{X,Y,Z,I\}$ (π pulses around all three axes) removes an *arbitrary* H_{SB}:

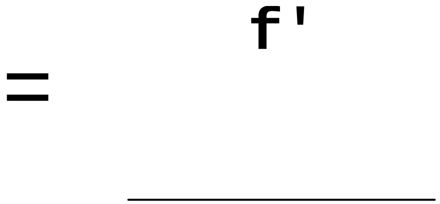
$$\mathtt{(XfX)(YfY)(ZfZ)(IfI)} = \underline{\mathtt{XfZfXfZf}}$$

Periodic DD: periodic repetition of the universal DD pulse sequence

The Effective Hamiltonian

Another view of the universal decoupling sequence:





$\mathbf{f} \equiv \exp[-i\tau H_{SB}]$

$\mathbf{f}' \equiv \exp[-iTH_{eff}(T)]$

$H_{eff}(T)=0$, ideally

The Effective Hamiltonian

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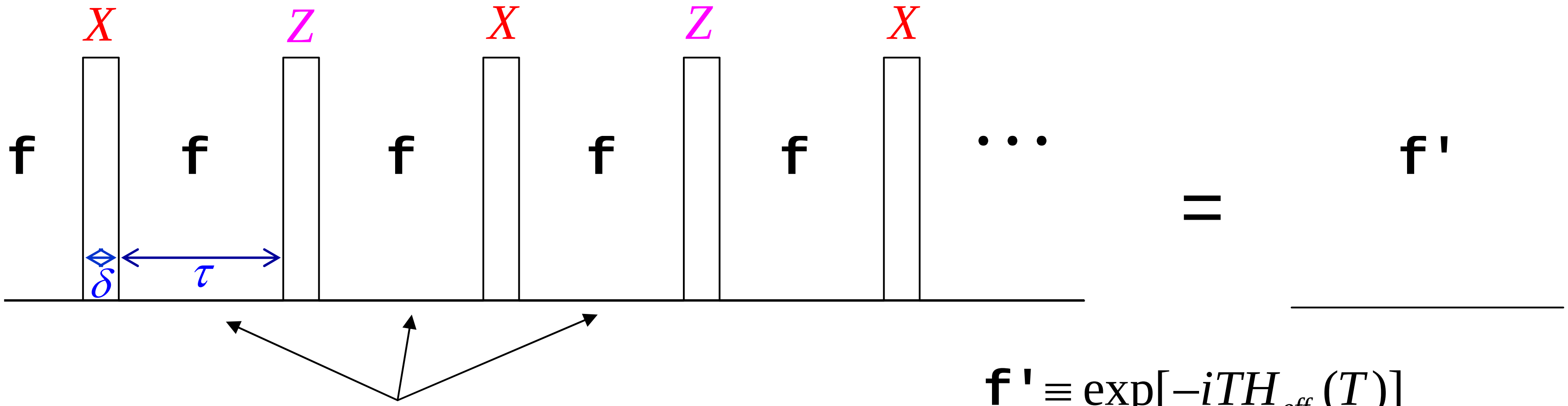

$\mathbf{f} \equiv \exp[-i\tau H_{SB}]$

$\mathbf{f'} \equiv \exp[-iTH_{eff}(T)]$

$H_{eff}(T) = 0$, ideally

But, errors accumulate…: $H_{eff}(T) \neq 0$

Periodic Dynamical Decoupling

PDD Strategy: repeat the basic XfZfZfXfZ cycle with total of N pulses.

The total duration is fixed at T. N can be changed.

Pulse interval: $\tau = T/N$

Define noise strength $\eta \equiv ||H_{\text{eff}}(T)||T$

norm of final effective system-bath Hamiltonian times the total duration.

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PDD leading order result for error:

$$\eta \propto N^{-1}$$

Can we do better?

Concatenated Universal Dynamical Decoupling

Nest the universal DD pulse sequence into its own free evolution periods **`f`** :

`p(1)= X f   Z f   X f   Z f`

[Khodjasteh & Lidar, PRL **95**, 180501 (2005)]

Concatenated Universal Dynamical Decoupling

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```
p(1)= X f   Z f   X f   Z f
p(2)= X p(1)Z p(1)X p(1)Z p(1)
      p(n+1)= X p(n)Z p(n)X p(n)Z p(n)
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[Khodjasteh & Lidar, PRL **95**, 180501 (2005)]

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```

Level	Concatenated DD Series after multiplying Pauli matrices
1	`XfZfXfZf`
2	`fZfXfZfYfZfXfZffZfXfZfYfZfXfZf`
3	`XfZfXfZfYfZfXfZffZfXfZfYfZfXfZfZfZfXfZfYfZfXfZffZfXfZfYfZfXfZfXfZf` `XfZfYfZfXfZffZfXfZfYfZfXfZfZfZfXfZfYfZfXfZffZfXfZfYfZfXfZf`

Length grows exponentially; how about error reduction?

[Khodjasteh & Lidar, PRL **95**, 180501 (2005)]

Performance of Concatenated Sequences

$$\text{error} \mapsto (\text{error})^2 \mapsto ((\text{error})^2)^2 \mapsto (((\text{error})^2)^2)^2 \mapsto \underset{k}{\cdots} \mapsto (\text{error})^{2^k}$$

[Khodjasteh & Lidar, PRA **75**, 062310 (2007)]

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$$\text{error} \mapsto (\text{error})^2 \mapsto ((\text{error})^2)^2 \mapsto (((\text{error})^2)^2)^2 \mapsto \underset{k}{\cdots} \mapsto (\text{error})^{2^k}$$

For fixed total time $T=N\tau$ and N zero-width (ideal) pulses:

$$\boxed{\eta \propto N^b N^{-c \log N}}$$

Compare to periodic DD: $\eta \propto N^{-1}$

[Khodjasteh & Lidar, PRA **75**, 062310 (2007)]

Part 2

Hybrid DD-DFS Quantum Gates

"Decouple while Compute"

arXiv:0911.2398

Computation

Problem: DD pulses interfere with computational pulses – they cancel everything!

How can they be reconciled?

- Need a commuting structure of pulses and computation.
- A solution:

  Use encoded qubits from a DFS.

  Logical gates over DFS generated by Heisenberg.

  DD pulses acting as global X and Z are still a universal decoupling group, and commute with Heisenberg

Heisenberg Computation over DFS is Universal

- Heisenberg exchange interaction:

$$H_{\text{Heis}} = \sum_{i,j} J_{ij}(X_iX_j + Y_iY_j + Z_iZ_j) \equiv \sum_{i,j} J_{ij}E_{ij}$$

- Universal over collective-decoherence DFS

[J. Kempe, D. Bacon, D.A.L., B. Whaley, Phys. Rev. A **63**, 042307 (2001)]

- Over 4-qubit DFS: $|0_L\rangle = \frac{1}{2}(|01\rangle - |10\rangle)(|01\rangle - |10\rangle)$

$$|1_L\rangle = \frac{1}{2\sqrt{3}}\left(2|0011\rangle + 2|1100\rangle - (|0110\rangle + |1001\rangle + |1010\rangle + |0101\rangle)\right)$$

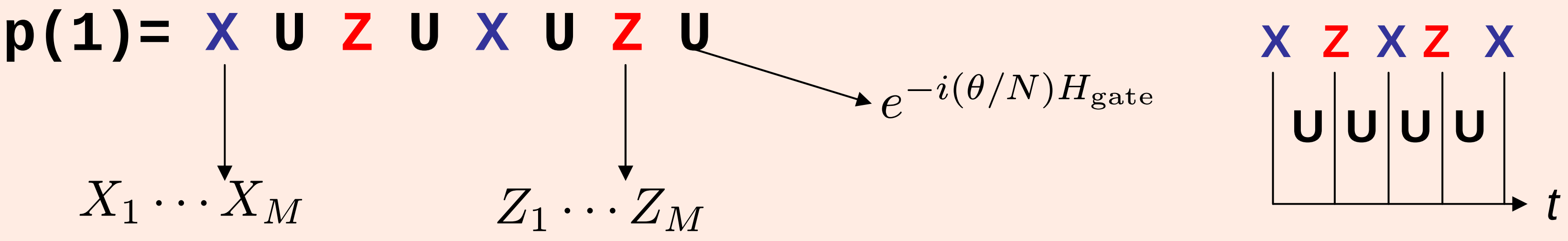
$$\bar{X} = -\tfrac{2}{\sqrt{3}}(E_{13} + \tfrac{1}{2}E_{12}) \qquad \bar{Z} = -E_{12}$$

$e^{i\theta\bar{X}}$ and $e^{i\theta\bar{Z}}$ generate arbitrary single encoded qubit gates

CNOT involves 42 elementary steps (D. Bacon, Ph.D. thesis)

Universal Decoupling Group Commutes with Heisenberg

- *n* levels of concatenation, $N=4^n$ pulses
- Universal decoupling group on *M* (even) system-spins:



p(1)= X U Z U X U Z U

$X_1 \cdots X_M$

$Z_1 \cdots Z_M$

$e^{-i(\theta/N)H_{\text{gate}}}$

p(2)= X p(1)Z p(1)X p(1)Z p(1) …

$$\boxed{[H_{\text{Heis}}, X \text{ or } Z] = 0}$$

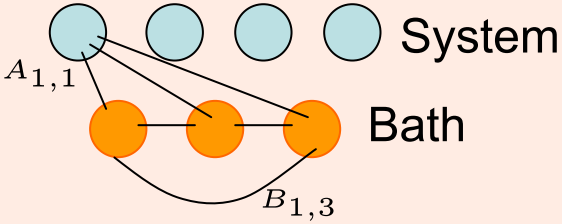
arXiv:0911.2398

Error Model: Electron qubits in GaAs, nuclear spin bath

- "Error Hamiltonian" (everything excluding DD):

$$H_e = \omega_S \sum_n \sigma_Z^{(n)} + \omega_B \sum_m \sigma_Z^{(m)} + \sum_{n<m} A_{n,m}\vec{\sigma}^{(n)} \cdot \vec{\sigma}^{(m)}$$

Heisenberg System-Bath (hyperfine coupling)

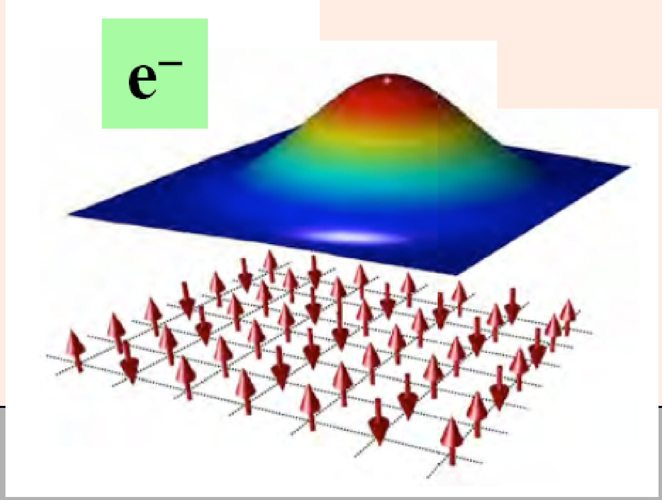


$$+ \sum_{n<m} B_{n,m}(\vec{\sigma}^{(n)} \cdot \vec{\sigma}^{(m)} - 3\sigma_X^{(n)}\sigma_X^{(m)})$$

Dipolar Bath-Bath

$A_{n,m} = C/2^{d_{n,m}}$, and $B_{n,m} = D/d_{n,m}^3$

β= $\sum_{n<m} B_{n,m} = 10\text{kHz}$



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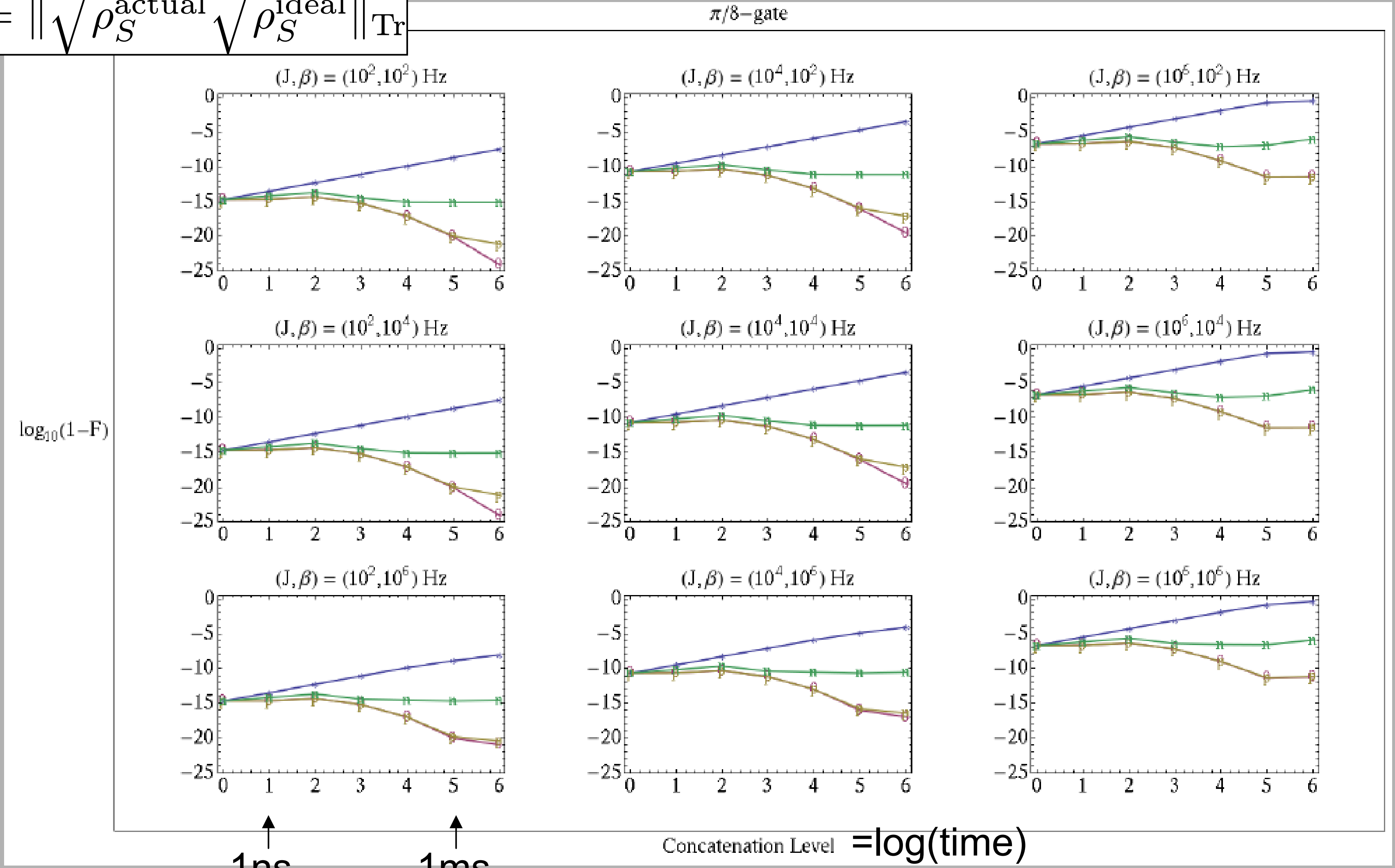
Model Assumptions for Exact Numerical Simulations

- Simulated discrete universal set: {π/8, Hadamard, CPhase}
- Single-encoded qubit gates: 4 system qubits, 6 bath qubits
- Encoded CPhase: 8 system qubits, 2 bath qubits
- Bath initialized in thermal equilibrium at zero K (equal superposition over all basis states)
- System initialized as $\frac{1}{\sqrt{2}}\left(\left|0_L\right\rangle+\left|1_L\right\rangle\right)$

Simulations for a protected logic gate: electron spin in a GaAs quantum dot

$$F = \|\sqrt{\rho_S^{\mathrm{actual}}}\sqrt{\rho_S^{\mathrm{ideal}}}\|_{\mathrm{Tr}}$$



=log(time)

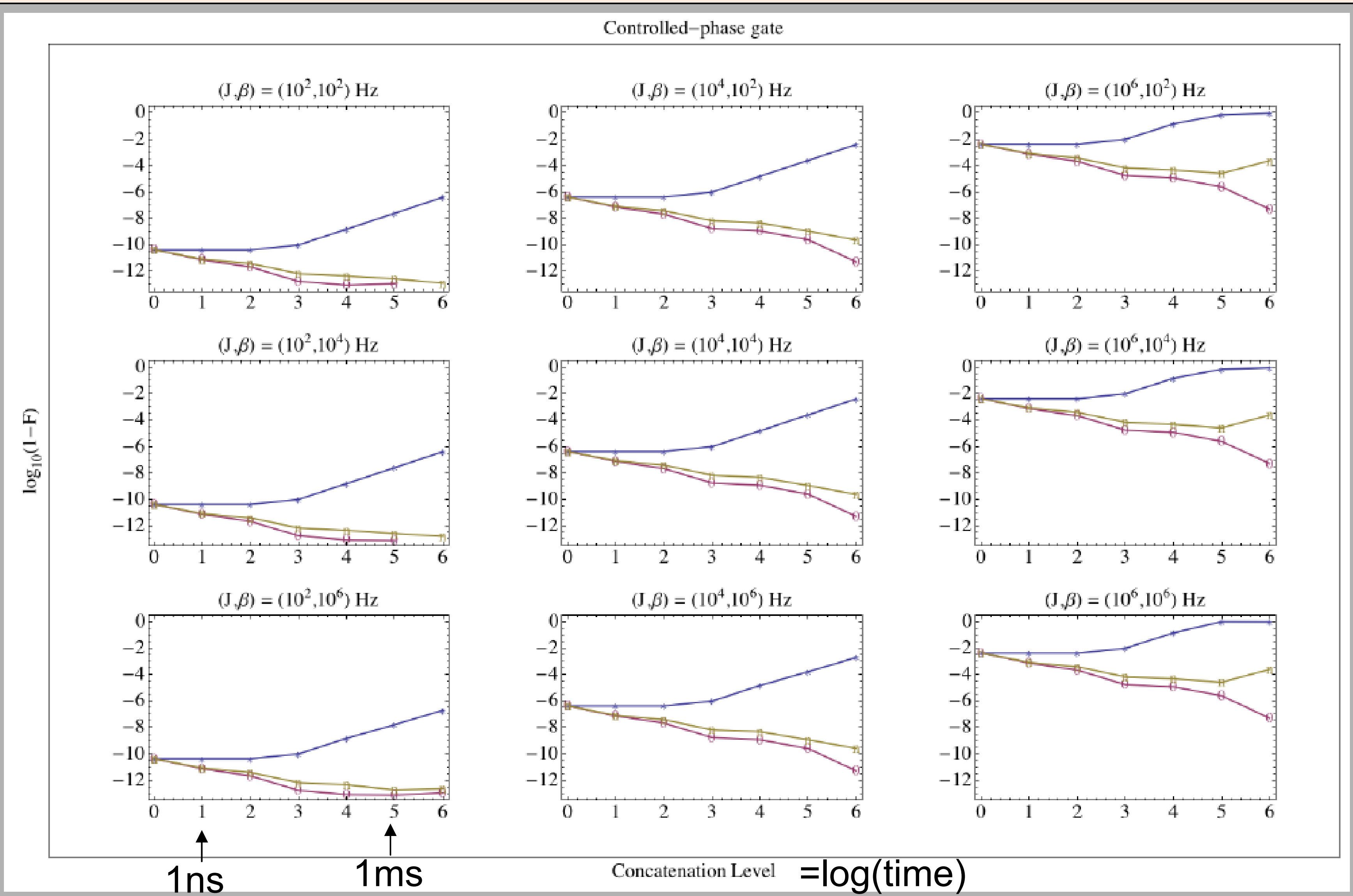
1ns

1ms

Gate implemented using Heisenberg interactions over 4 qubit DFS code.
Pulse widths: 0, 1ps, 1ns

arXiv:0911.2398

Simulations for a protected CPHASE gate: electron spin in GaAs quantum dot



Gate implemented using Heisenberg interactions over two 4 qubit DFS codes. Pulse widths: 0, 1ps

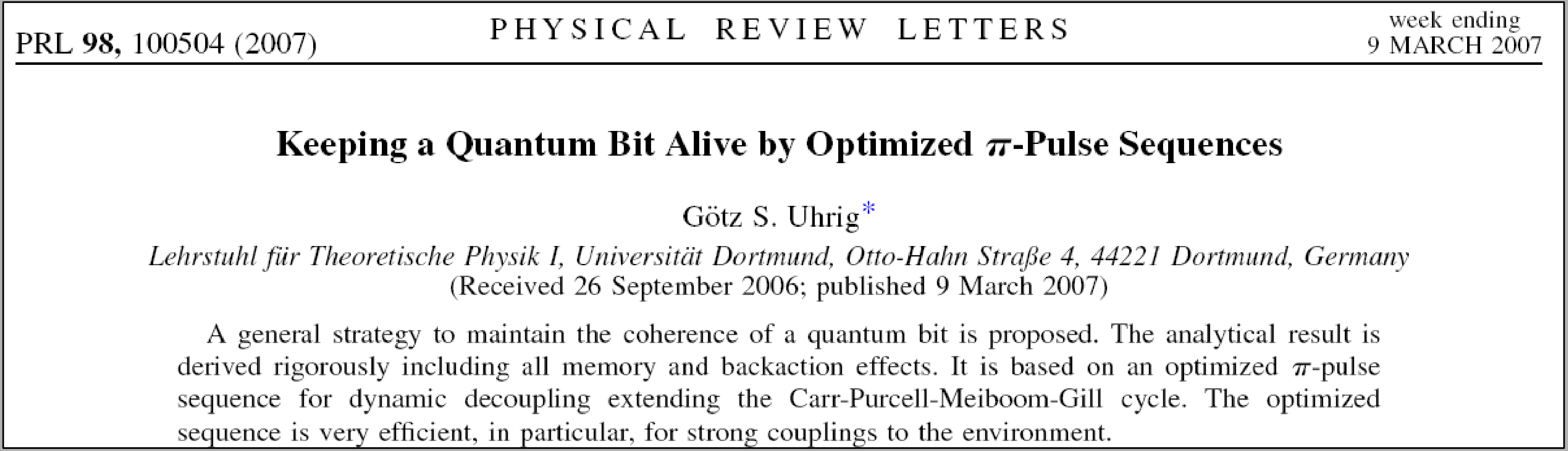
arXiv:0911.2398

Better than Concatenated DD?

Does there exist an optimal pulse sequence?

Optimal = removes maximum decoherence
with least possible number of pulses

Better than Concatenated DD?



PRL **98,** 100504 (2007) PHYSICAL REVIEW LETTERS week ending 9 MARCH 2007

Keeping a Quantum Bit Alive by Optimized π-Pulse Sequences

Götz S. Uhrig*

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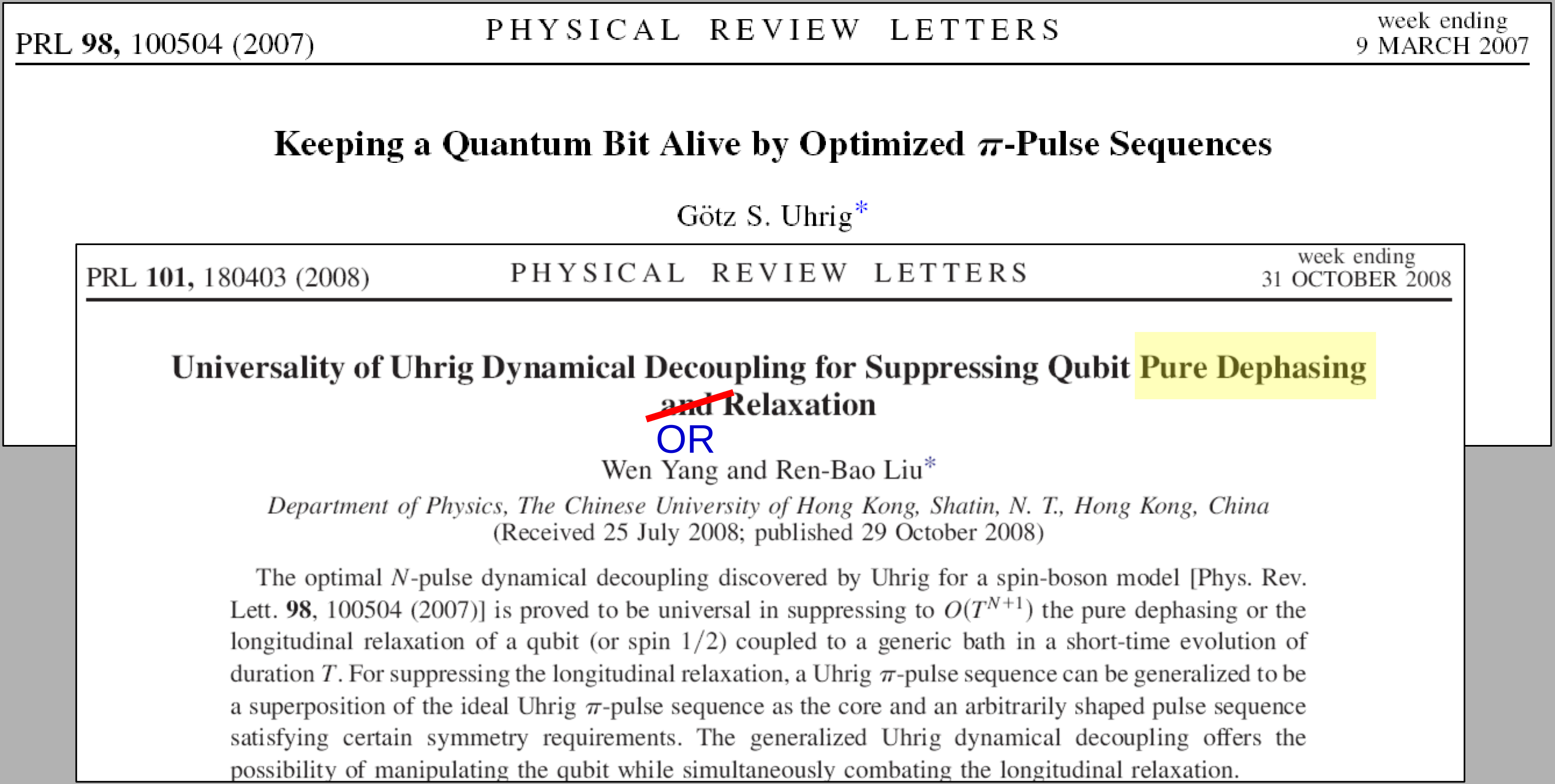
(Received 26 September 2006; published 9 March 2007)

A general strategy to maintain the coherence of a quantum bit is proposed. The analytical result is derived rigorously including all memory and backaction effects. It is based on an optimized π-pulse sequence for dynamic decoupling extending the Carr-Purcell-Meiboom-Gill cycle. The optimized sequence is very efficient, in particular, for strong couplings to the environment.

Better than Concatenated DD?



PRL **98,** 100504 (2007) PHYSICAL REVIEW LETTERS week ending 9 MARCH 2007

Keeping a Quantum Bit Alive by Optimized π-Pulse Sequences

Götz S. Uhrig*

PRL **101,** 180403 (2008) PHYSICAL REVIEW LETTERS week ending 31 OCTOBER 2008

Universality of Uhrig Dynamical Decoupling for Suppressing Qubit Pure Dephasing ~~and~~ Relaxation

OR

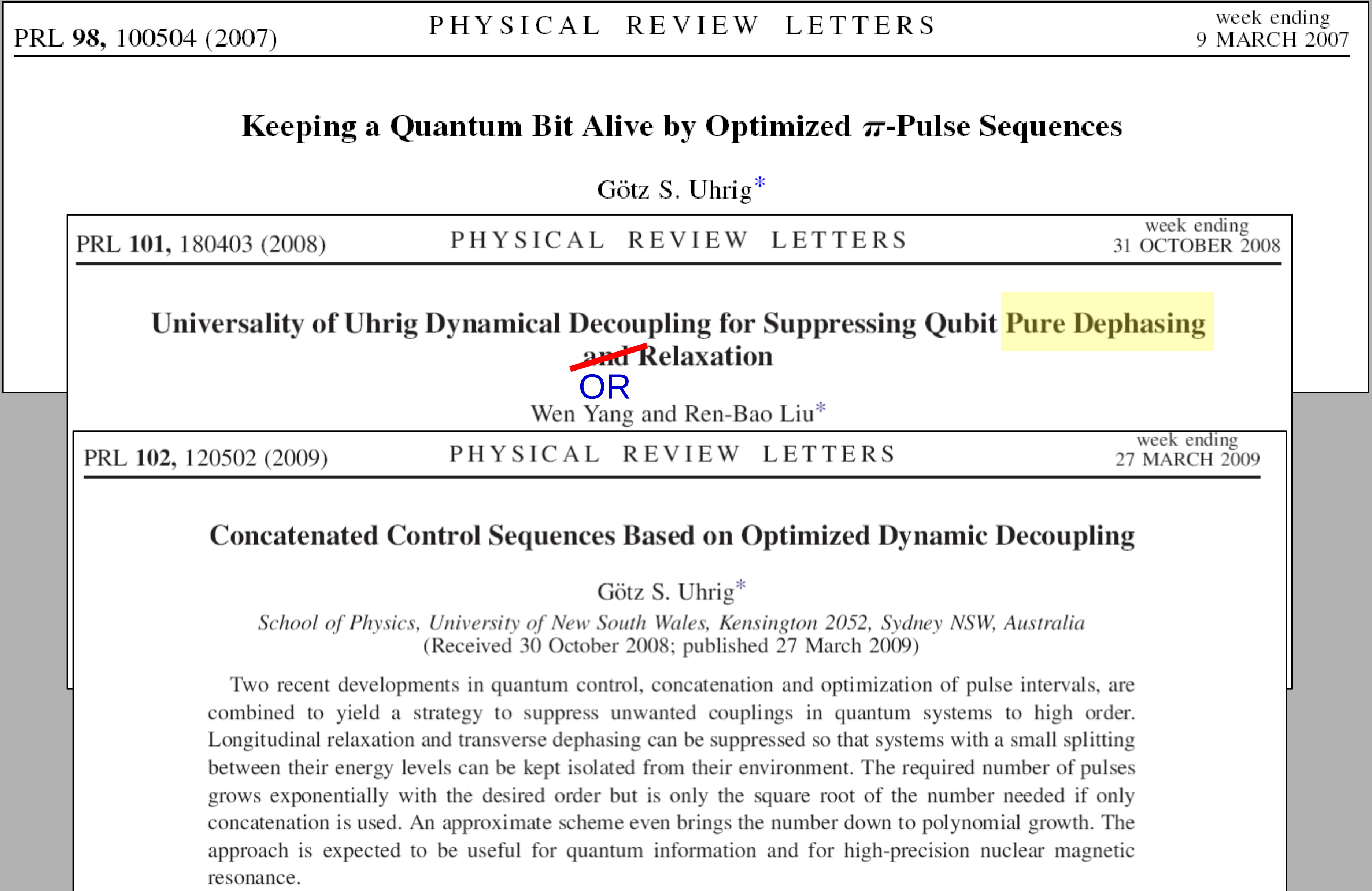
Wen Yang and Ren-Bao Liu*

Department of Physics, The Chinese University of Hong Kong, Shatin, N. T., Hong Kong, China

(Received 25 July 2008; published 29 October 2008)

The optimal N-pulse dynamical decoupling discovered by Uhrig for a spin-boson model [Phys. Rev. Lett. **98**, 100504 (2007)] is proved to be universal in suppressing to $O(T^{N+1})$ the pure dephasing or the longitudinal relaxation of a qubit (or spin 1/2) coupled to a generic bath in a short-time evolution of duration T. For suppressing the longitudinal relaxation, a Uhrig π-pulse sequence can be generalized to be a superposition of the ideal Uhrig π-pulse sequence as the core and an arbitrarily shaped pulse sequence satisfying certain symmetry requirements. The generalized Uhrig dynamical decoupling offers the possibility of manipulating the qubit while simultaneously combating the longitudinal relaxation.

Better than Concatenated DD?



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Universality of Uhrig Dynamical Decoupling for Suppressing Qubit Pure Dephasing ~~and~~ Relaxation

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Wen Yang and Ren-Bao Liu*

PRL **102,** 120502 (2009) PHYSICAL REVIEW LETTERS week ending 27 MARCH 2009

Concatenated Control Sequences Based on Optimized Dynamic Decoupling

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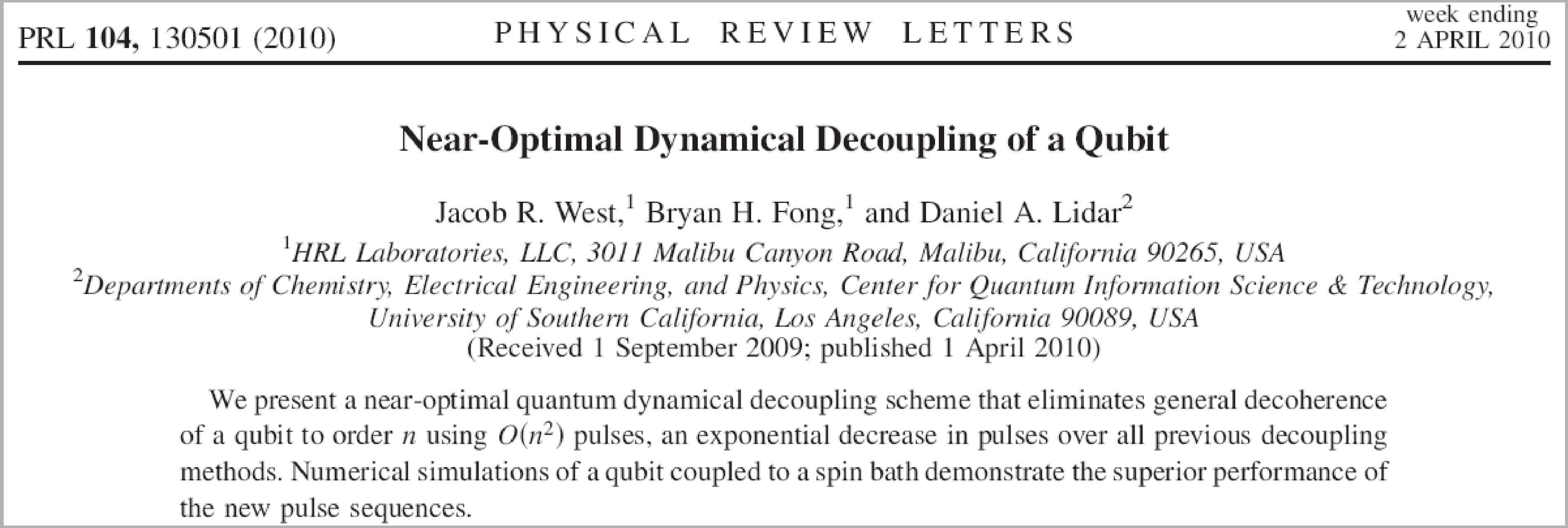
School of Physics, University of New South Wales, Kensington 2052, Sydney NSW, Australia

(Received 30 October 2008; published 27 March 2009)

Two recent developments in quantum control, concatenation and optimization of pulse intervals, are combined to yield a strategy to suppress unwanted couplings in quantum systems to high order. Longitudinal relaxation and transverse dephasing can be suppressed so that systems with a small splitting between their energy levels can be kept isolated from their environment. The required number of pulses grows exponentially with the desired order but is only the square root of the number needed if only concatenation is used. An approximate scheme even brings the number down to polynomial growth. The approach is expected to be useful for quantum information and for high-precision nuclear magnetic resonance.

Better than Concatenated DD?



PRL **104,** 130501 (2010) PHYSICAL REVIEW LETTERS week ending 2 APRIL 2010

Near-Optimal Dynamical Decoupling of a Qubit

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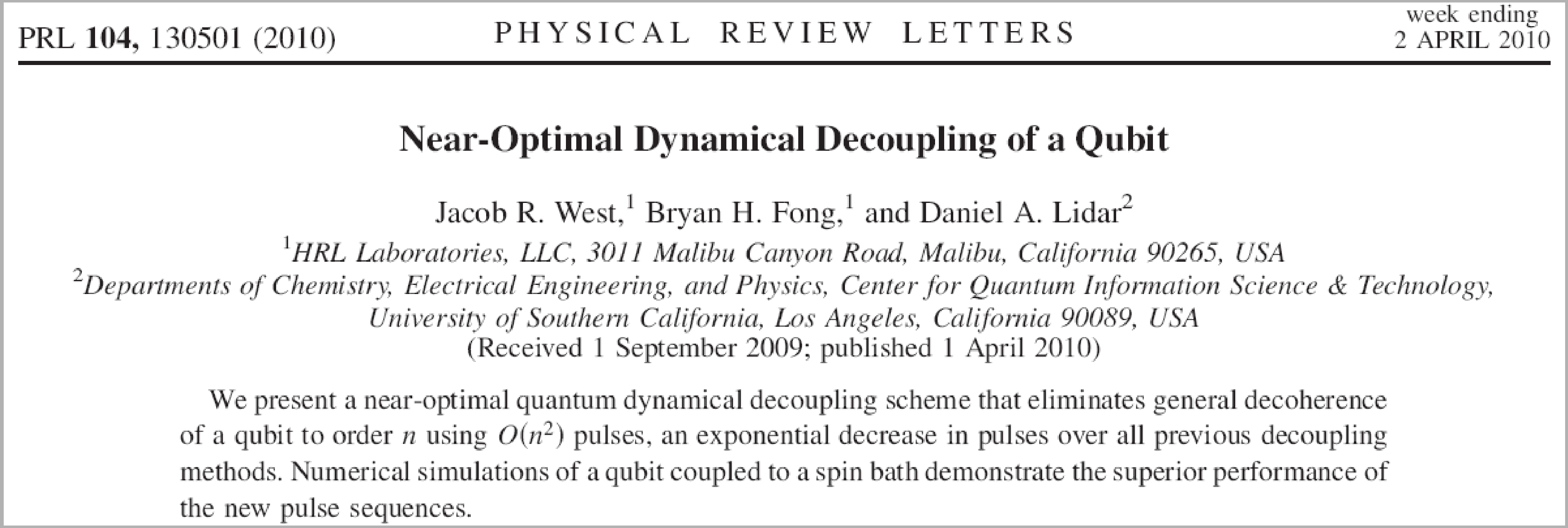
(Received 1 September 2009; published 1 April 2010)

We present a near-optimal quantum dynamical decoupling scheme that eliminates general decoherence of a qubit to order n using $O(n^2)$ pulses, an exponential decrease in pulses over all previous decoupling methods. Numerical simulations of a qubit coupled to a spin bath demonstrate the superior performance of the new pulse sequences.

Better than Concatenated DD?



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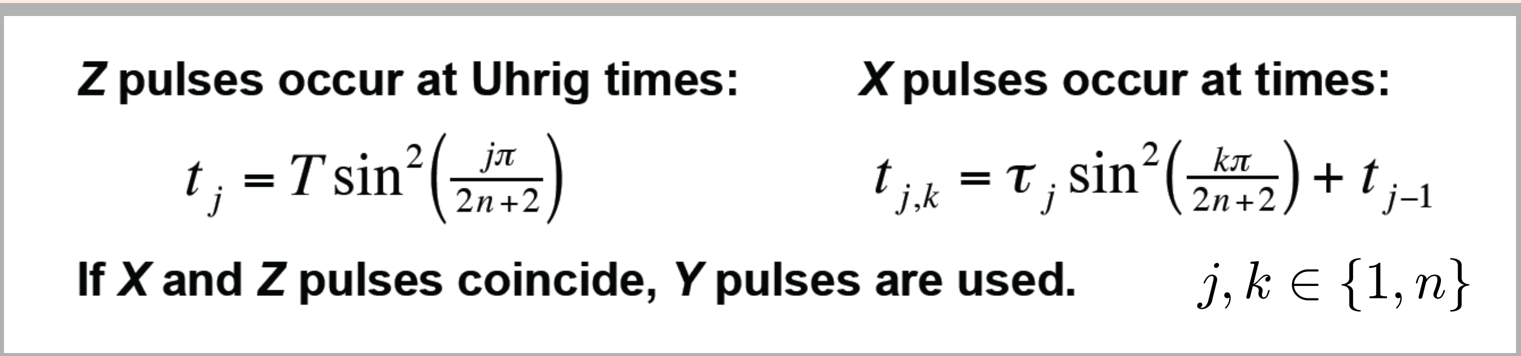
“Quadratic DD” eliminates the first *n* orders in the Dyson series of the joint system-bath propagator using n^2 pulses

Concatenated DD requires 4^n pulses to do the same, approximately

Inner workings of Quadratic DD



Z **pulses occur at Uhrig times:**

$$t_j = T\sin^2\left(\frac{j\pi}{2n+2}\right)$$

X **pulses occur at times:**

$$t_{j,k} = \tau_j \sin^2\left(\frac{k\pi}{2n+2}\right) + t_{j-1}$$

If *X* and *Z* pulses coincide, *Y* pulses are used.

$$j, k \in \{1, n\}$$

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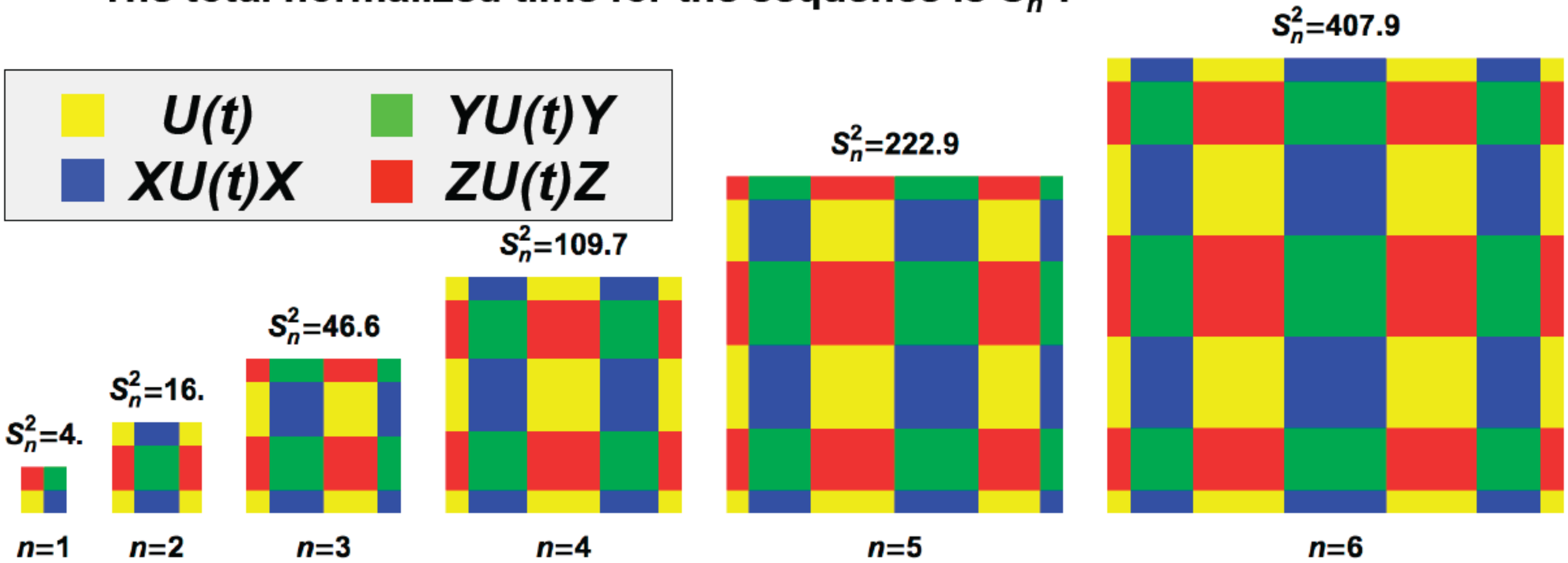
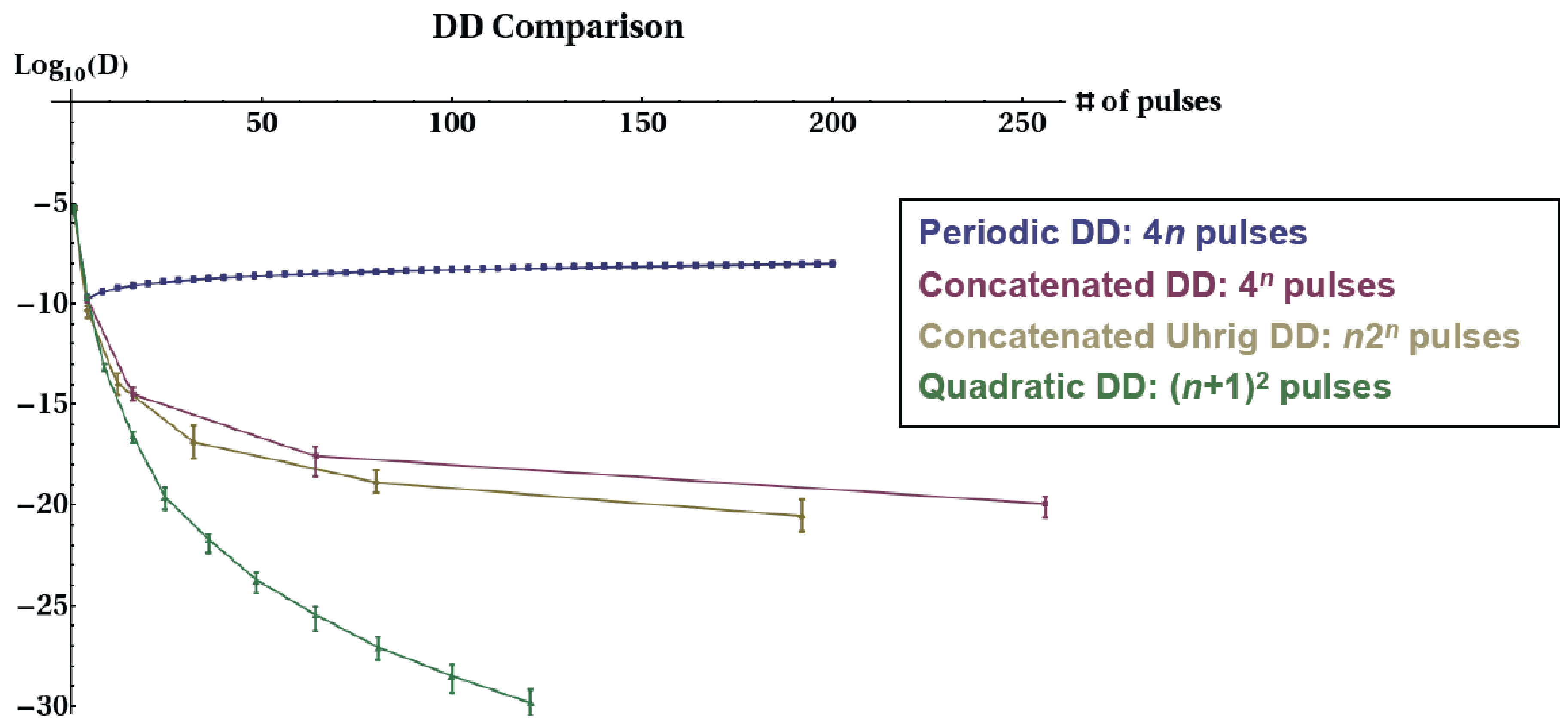
- **This can be visualized as an outer product of Uhrig sequences:**
  - **Colors correspond to interval conjugation type (*I*,*X*,*Y*,*Z*)**
  - **Areas correspond to interval duration**
  - **Final pulse sequence is read off row-by-row down the 2D array**
- **The total normalized time for the sequence is S_n^2.**



For every value of n, the first $\sqrt{n}$ terms in the Dyson series are removed

Comparison of DD Sequences



$$H = \beta(I \otimes B_I) + J(X \otimes B_X + Y \otimes B_Y + Z \otimes B_Z)$$

$$B_\alpha = \sum_{i\neq j}\sum_{k,l=0}^{3} r_{kl}^{\alpha}(\sigma_{i,k} \otimes \sigma_{j,l})$$

$J\tau = \beta\tau = 10^{-6}$. The shortest pulse interval τ is the same in all simulations

Use QDD in computation? Open.

Part 3: Decouple-then-compute

Or: how dynamical decoupling can help fault tolerance

arXiv:0911.3202

DL, with Hui-Khoon Ng and John Preskill

IQI-Caltech, CQIST-USC

Obstacles to scalable quantum computation

- Environment causes decoherence.
- Gates for computation are noisy.
- Gates for quantum error correction are noisy

Is reliable quantum computation nevertheless possible?

Yes, under certain reasonable noise models:

the *quantum accuracy threshold theorem*

Noise Model

- System and bath are coupled via an “error” Hamiltonian H_{err}.
- Time between gates is τ_0.
- The *noise strength* is $\eta \equiv \|H_{\mathrm{err}}\|\tau_0$.

Quantum accuracy threshold theorem

- Simulate ideal L-gate circuit using an L^*-gate circuit protected by many levels of concatenated quantum error correction. There exists a *threshold* η_0 such that concatenation suffices for accurate simulation of arbitrarily long quantum computations, if the noise is “sufficiently weak”:

$$\eta < \eta_0 \sim 10^{-4}$$

- η also determines number of extra gates needed to compute fault-tolerantly:

$$L^* \propto L \left(\frac{\log(\eta_0 L/\varepsilon)}{\log(\eta_0/\eta)} \right)^c$$

(ε is desired accuracy error (upper bound on trace norm difference), $c = O(1)$, a circuit dependent constant)

Let's add dynamical decoupling (DD): each gate will be preceded by a DD pulse sequence. What do we stand to gain?

- DD doesn't involve measurements
- DD doesn't require ancillas

$\Longrightarrow$ DD easier to implement than QEC.

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  $H_{\text{err}} \longmapsto H_{\text{eff}}$, hence
  $\eta = \|H_{\text{err}}\|\tau_0 \longmapsto \eta_{\text{DD}} = \|H_{\text{eff}}\|\tau_0$

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  $\eta = \|H_{\mathrm{err}}\|\tau_0 \longmapsto \eta_{\mathrm{DD}} = \|H_{\mathrm{eff}}\|\tau_0$
- Can adding DD to fault-tolerant circuit weaken noise strength η and reduce resource requirements?
  I.e., can we have $\eta_{\mathrm{DD}} < \eta$?
  And if we did, what good what it do?

- Gates *un*protected by DD: scalable if the noise strength of unprotected gates is below the accuracy threshold, $\eta_{\text{noDD}} \equiv \eta < \eta_0$.
- Gates protected by DD: scalable if $\eta_{\text{DD}} < \eta_0$.

Two ways DD can help FT:

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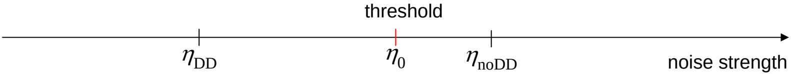
- 

  Arbitrarily large quantum circuits can be simulated accurately with DD-protected gates, but not with unprotected gates

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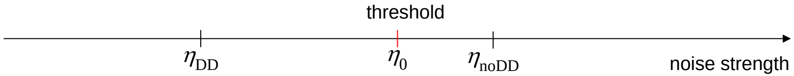
- 

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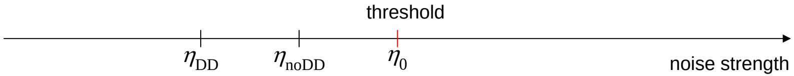
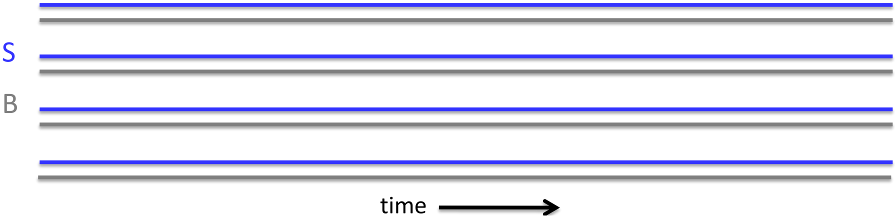
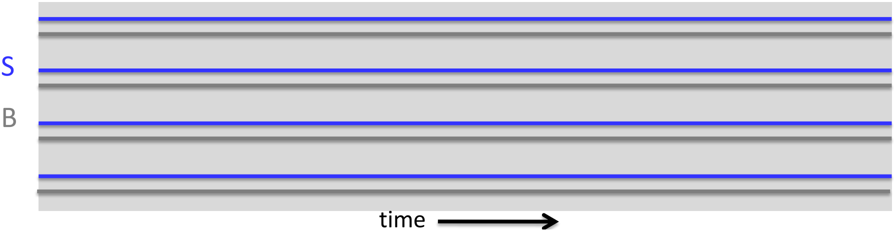
- 

  Even when $\eta_{\mathrm{noDD}} < \eta_0$, DD may reduce the overhead cost of fault-tolerant quantum computing if $\eta_{\mathrm{DD}} < \eta_{\mathrm{noDD}}$

The task

- Identify and compute the noise strength of the DD-protected circuit, η_{DD}
- Identify conditions under which $\eta_{\mathrm{DD}} < \eta_{\mathrm{noDD}}$
- Determine which DD schemes are best at lowering the noise strength while being compatible with QEC

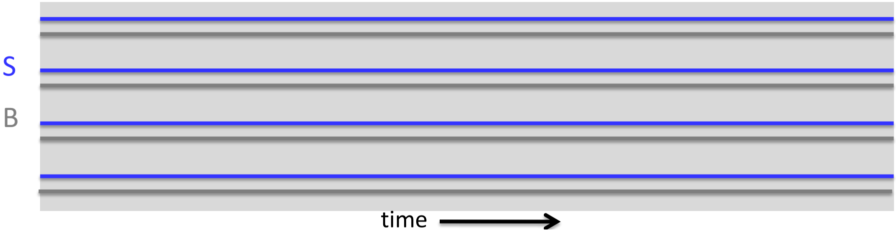

S
B
time



Joint evolution of S and B; noise Hamiltonian H.

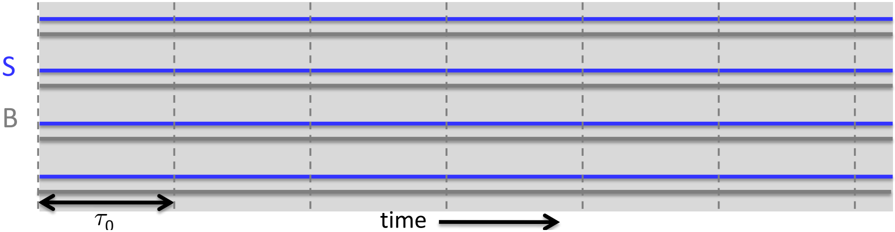
$$\begin{aligned} H_B &\equiv \text{free bath evolution} \\ \left.\begin{aligned} H_S^0 &\equiv \text{free no-noise system evolution} \\ H_{SB} &\equiv \text{system-bath interaction} \end{aligned}\right\} & H_{\text{err}} \equiv H_S^0 + H_{SB} \end{aligned}$$

$$H_{\text{full}} = H = H_B + H_{\text{err}}.$$



Add computation:

- Sequence of gates used to fault-tolerantly simulate ideal circuit.



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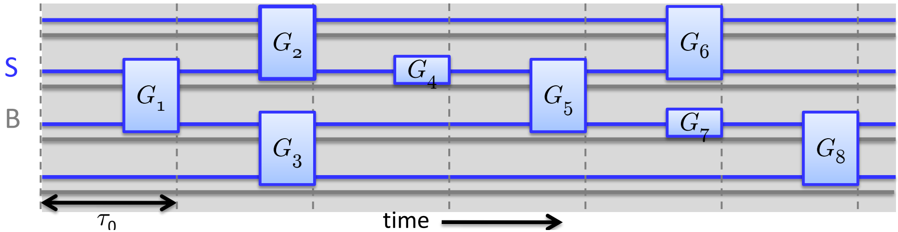

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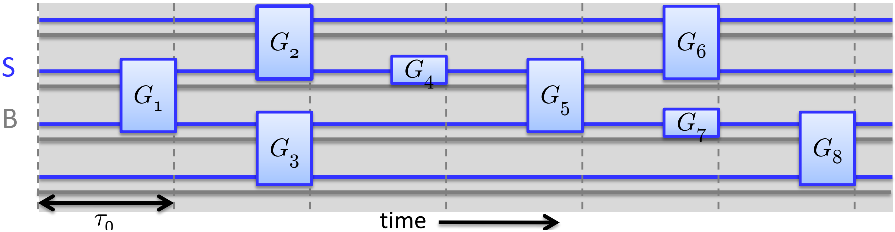

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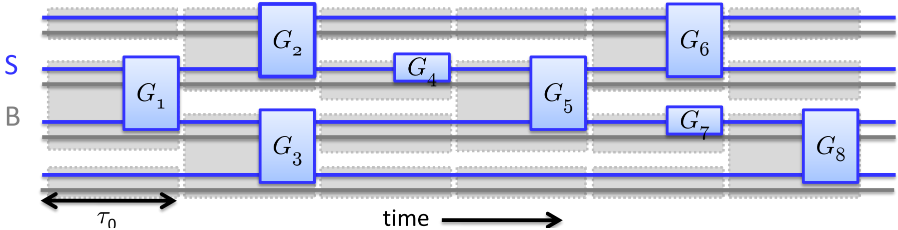
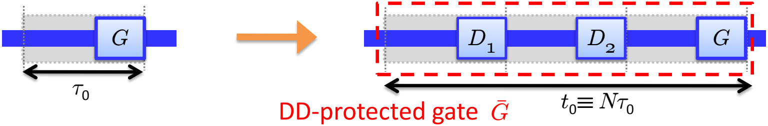
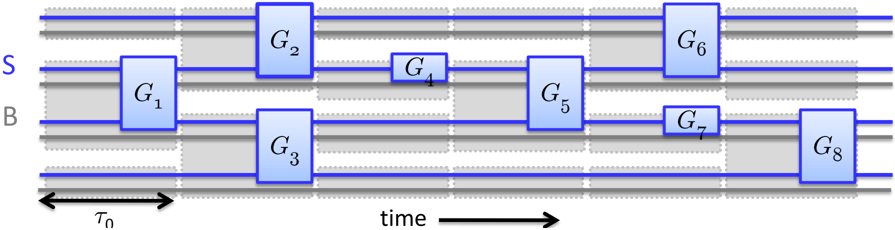
$$H_{\mathrm{full}} = H + H_c(t) = H_B + H_{\mathrm{err}} + H_c(t).$$



Add DD:

- Replace each gate by (DD pulse sequence + gate)





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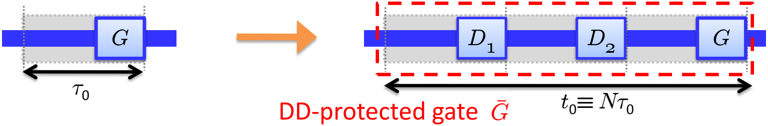

$$H_{\text{full}} = H + H_c(t), \quad H_c(t) \text{ includes DD pulses.}$$

Noise strength of a DD-protected gate

Recall: gate G; DD-protected gate $\overline{G}$; t_0 time taken by $\overline{G}$.

- Local-bath assumption – examine each DD-protected gate separately.
- Noise strength η^0_{DD} – deviation from ideal evolution ($H_{\mathrm{err}} = 0$):

$$\eta^0_{\mathrm{DD}} \equiv max_a \| \overline{G}_a - G_a \otimes U_B(t_0) \|$$

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- $\eta^0_{\mathrm{DD}} \leq \|\underbrace{\Omega_1(t_0) + it_0 H_B}_{\Omega_1'(t_0)}\| + \sum_{n=2}^{\infty} \|\Omega_n(t_0)\|$.
  - $\Omega_n(t_0)$ are terms in Magnus expansion for $\overline{G}$ in "toggling frame".
  - Computed from toggling frame Hamiltonian $\tilde{H}(t) \equiv U_c^\dagger(t) H U_c(t)$.
  - First-order decoupling: $\Omega_1'(t_0) = 0$. Assumed for non-trivial DD with zero-width pulses.

Bound on the noise strength

- Compute bounds on individual Magnus terms.
- Parameters: $J\tau_0, \epsilon\tau_0 \ll 1$
  - $J \equiv \|H_{\text{err}}\|$,
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- Noise strength for DD-protected gate:

$$\boxed{\eta_{\text{DD}} \equiv (Jt_0) \sum_{n=1}^{5} C_n (\epsilon t_0)^{n-1}}$$

	General	**Time-symmetric**
C_1	0	0
C_2	1	0
C_3	$4/9$	$2/9$
C_4	$11/9$	0
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- Time-symmetric DD sequence: $\tilde{H}(t_0 - t) = \tilde{H}(t)$
  - $\Omega_n(t_0) = 0$ for all n even.
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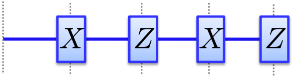
- DD weakens noise strength if $\eta_{\text{DD}} < \eta_{\text{noDD}} = J\tau_0$.
  - Less stringent fault tolerance threshold condition $\eta_{\text{DD}} < \eta_0$.
  - Lower resource requirements to attain same level of accuracy.

Example – single-qubit errors (*i*: system qubit label).

$$H = H_B + H_{\text{err}}, \quad \text{with } H_{\text{err}} \equiv \sum_{i,\alpha} \sigma_\alpha^{(i)} \otimes B_\alpha^{(i)}.$$

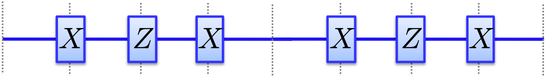
- Universal decoupling sequence:



$t_0 = 4\tau_0$

$$\eta_{\text{DD}} \sim (Jt_0)(\epsilon t_0) = (4J\tau_0)(4\epsilon\tau_0)$$

- Time-symmetric version: Apply universal decoupling sequence, repeat, but time-reversed.

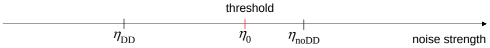


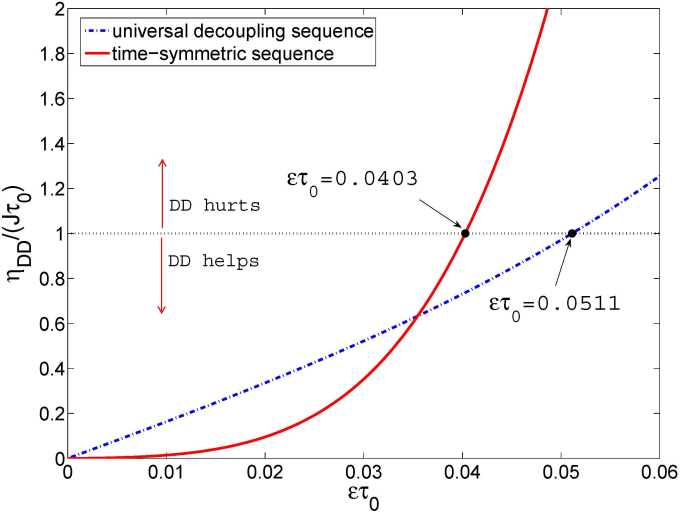
$t_0 = 8\tau_0$

$$\eta_{\text{DD}} \sim \frac{2}{9}(Jt_0)(\epsilon t_0)^2 = \frac{2}{9}(8J\tau_0)(8\epsilon\tau_0)^2$$

- Recall: $\eta_{\text{noDD}} = J\tau_0$.

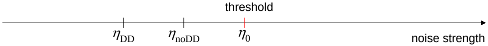
Effective noise strengths for zero-width pulses

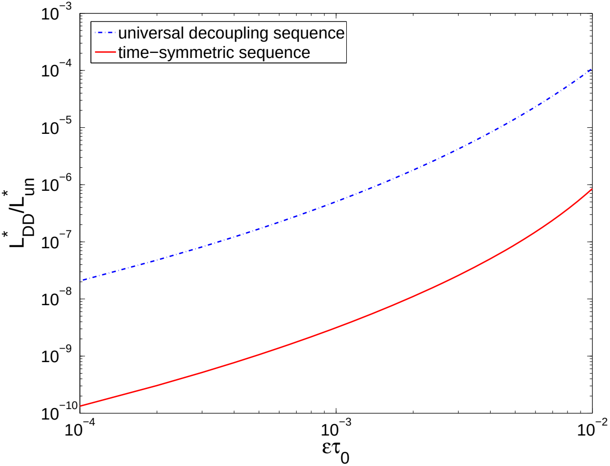




Reduction in number of gates required in fault tolerant simulation

$J\tau_0 = 10^{-4}$, N zero-width pulses, L^*_{DD} gates for simulation with DD, L^*_{un} without DD, $\frac{L^*_{\mathrm{DD}}}{L^*_{\mathrm{un}}} = N\left(\frac{\log(\eta_0/\eta_{\mathrm{noDD}})}{\log(\eta_0/\eta_{\mathrm{DD}})}\right)^c$





Concatenated DD

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Concatenated DD sequence is recursively generated.
E.g., for single-qubit universal decoupling sequence:

$$p_1 = ZIXIZIXI$$

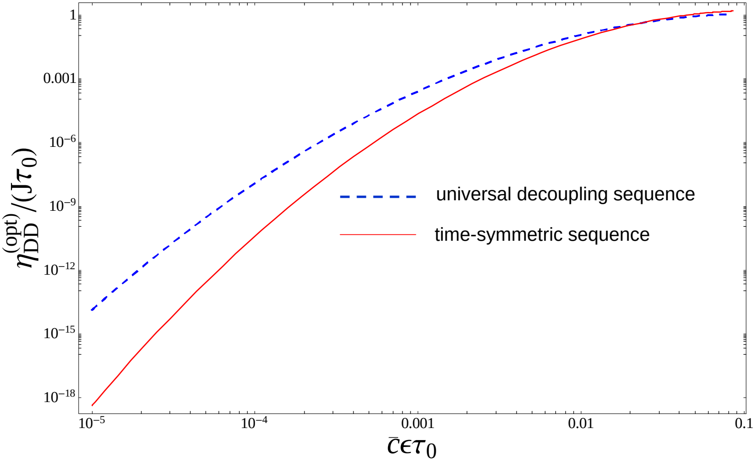
$$p_2 = Zp_1Xp_1Zp_1Xp_1$$

$$p_k = Zp_{k-1}Xp_{k-1}Zp_{k-1}Xp_{k-1}$$

- Such sequences don't allow errors to build up by reducing them at all concatenation levels
- However, if sequence becomes too long errors build up and effectiveness diminishes
- There exists an optimal concatenation level:

$$k_{\rm opt} = -\log(\epsilon\tau_0) + \text{const}$$

- universal DD: $4N\delta J \leq \frac{\eta_{\mathrm{CDD}}}{J\tau_0} \leq \frac{1}{N}(\bar{c}\epsilon\tau_0)^{-\frac{1}{2}\log_N(\bar{c}\epsilon\tau_0)-\frac{3}{2}}$ ($\bar{c}$ is a constant, δ is pulse width)
- time-symmetric DD: $8N\delta J \leq \frac{\eta_{\mathrm{CDD}}}{J\tau_0} \leq \frac{1}{N^{3/4}}(\bar{c}\epsilon\tau_0)^{-\log_N(\bar{c}\epsilon\tau_0)-2}$



Conclusions

Obtained rigorous bounds obtained on the Magnus expansion.
Enabled answering:

Can adding DD to fault-tolerant circuit weaken the noise strength η and reduce resource requirements?

- Yes, if $\epsilon\tau_0$ is small enough.
- Time-symmetric sequence: extra power of $\epsilon\tau_0$ in noise strength. Can result in smaller noise strength if sequence is not too long.
- Concatenated sequence: orders of magnitude reduction of noise strength as $\epsilon\tau_0$ shrinks.
- Result: can enable fault tolerant quantum computation even when unprotected gates are above threshold.