

# *Tutorial:*

# *Quantum Information & Cold Atoms*

- cold atoms as a tool in quantum information
- applications:
  - quantum computing
  - quantum communications
  - precision measurements

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# Entangled States

- entanglement



states:  $|0\rangle \otimes |0\rangle$

$|1\rangle \otimes |1\rangle$

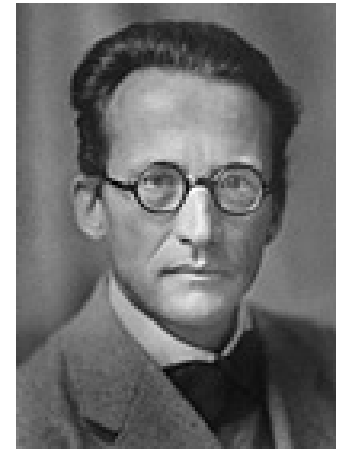
... product states

but also ...

$\frac{1}{\sqrt{2}} (|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)$  ... entangled

- fundamental aspects of quantum mechanics
  - incompatibility of QM with LHVT
  - decoherence
  - measurement theory (?)
- applications
  - quantum communications & computing
  - precision measurement

Schrödinger:  
Verschränkung



# Engineering *Entangled States*

We need ...

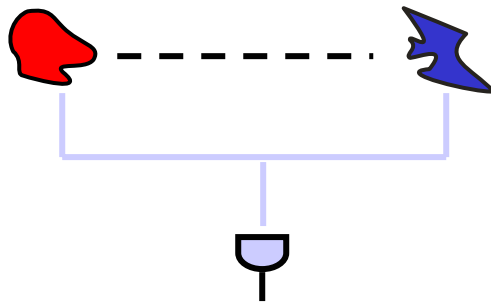
- “quantum engineering”



$$|a\rangle_A |b\rangle_B \rightarrow \sum c_{ab} |a\rangle_A |b\rangle_B$$

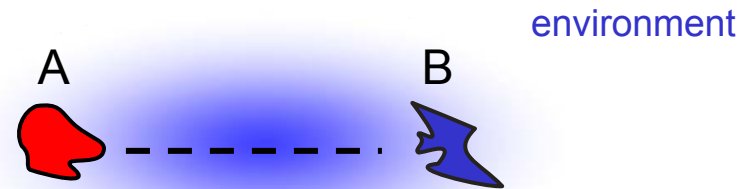
Hamiltonian evolution

- or: “quantum gambling”



measurement

- isolation



$$|\phi\rangle_A |\phi\rangle_B |E\rangle \rightarrow |\Psi\rangle_{ABE}$$

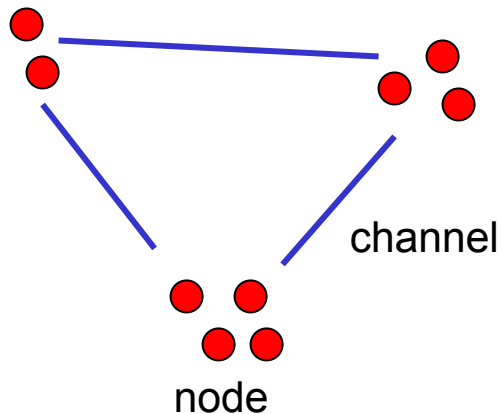
$$\rho_{AB} = \text{tr}_E |\Psi\rangle_{ABE} \langle \Psi|$$

$$\neq |\Psi\rangle_{AB} \langle \Psi|$$

Quantum optical systems provide one of the best set-ups to create entangled states in a controlled way.

# Quantum Optics Road Map

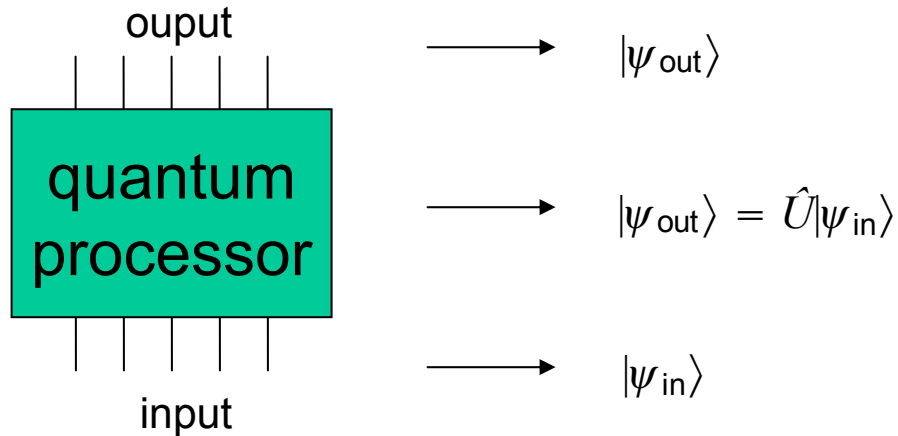
- Starting from the quantum optical tool box:
  - ✓ photons
  - ✓ atoms... manipulate & couple on level of single quanta
- we assemble large systems ...



- **Nodes: local quantum computing**
  - store quantum information
  - local quantum processing
- **Channels: quantum communication**
  - transmit quantum information

# Quantum information processing

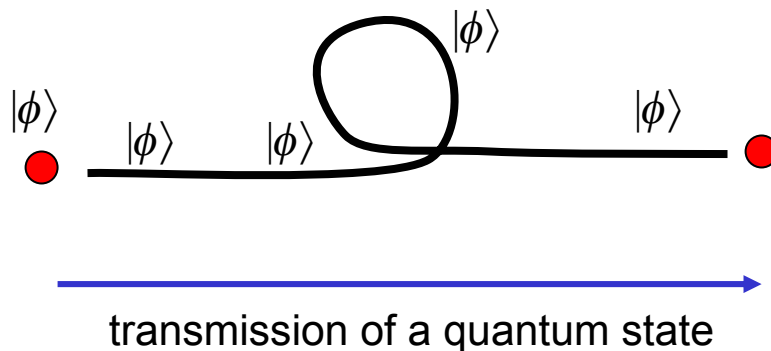
- quantum computing



quantum weirdness:

- ✓ superposition
- ✓ entanglement
- ✓ interference
- ✓ nonclonability and uncertainty
- ✓ no decoherence!

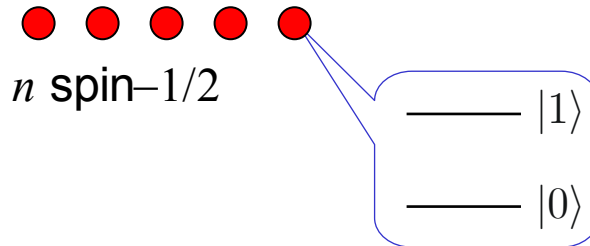
- quantum communications



- ✓ teleportation
- ✓ cryptography

# Qubits, Quantum Gates etc.

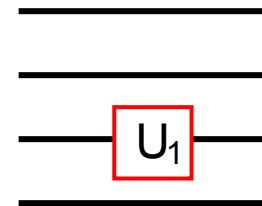
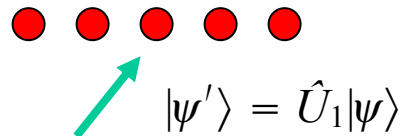
- quantum bits or qubits



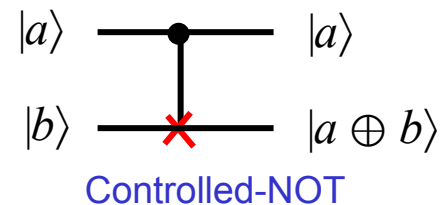
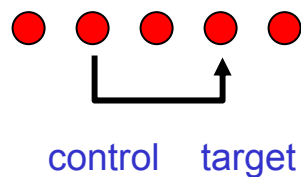
example: two qubits

$$|\psi\rangle = c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle$$

- single qubit gate



- two qubit gate



## *1. Cold atoms as quantum memory*

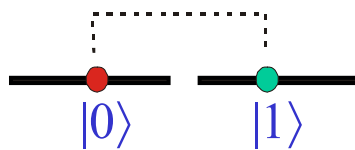
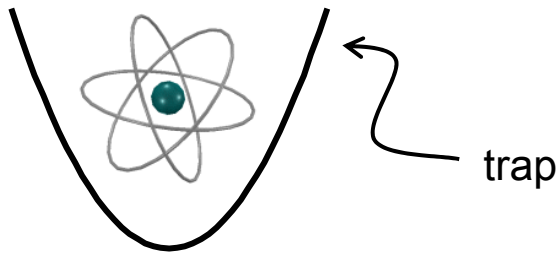
- technique; cooling and trapping of atoms
- decoherence properties

## *2. How to entangle atoms?*

# 1. Cold atoms as quantum memory

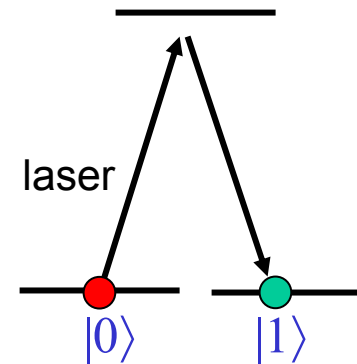
- cold atoms

single trapped atom:

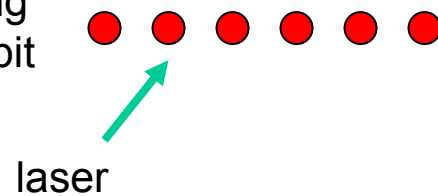


qubit in *longlived*  
internal states

- single qubit gates

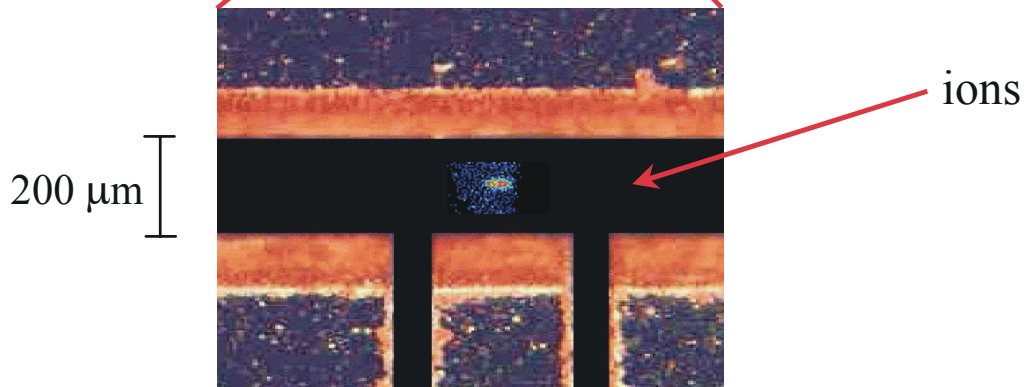
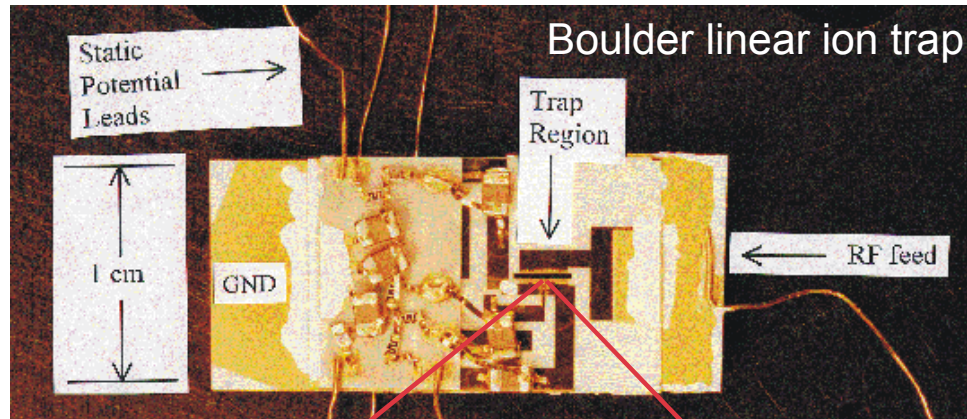


addressing  
single qubit



## Remarks: Traps ...

- ion traps



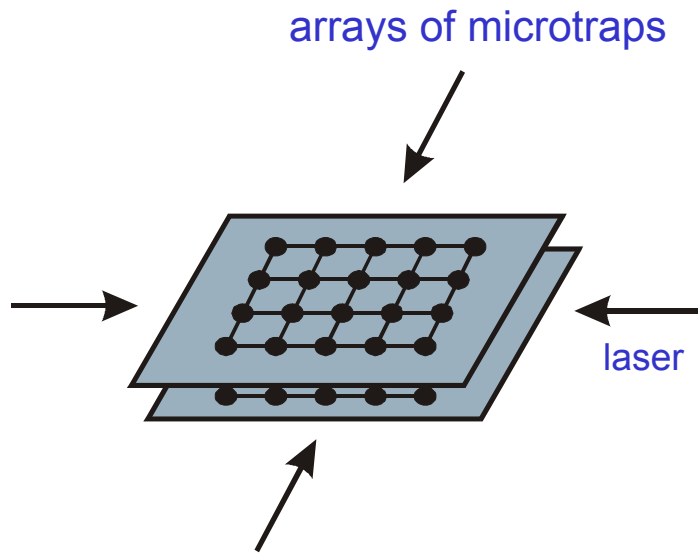
issues:

- ✓ conservative potential (heating?)
- ✓ single atom loading
- ✓ laser cooling to ground state

NIST Boulder, Innsbruck, Munich,  
Hamburg, Aarhus, Oxford, London, ...

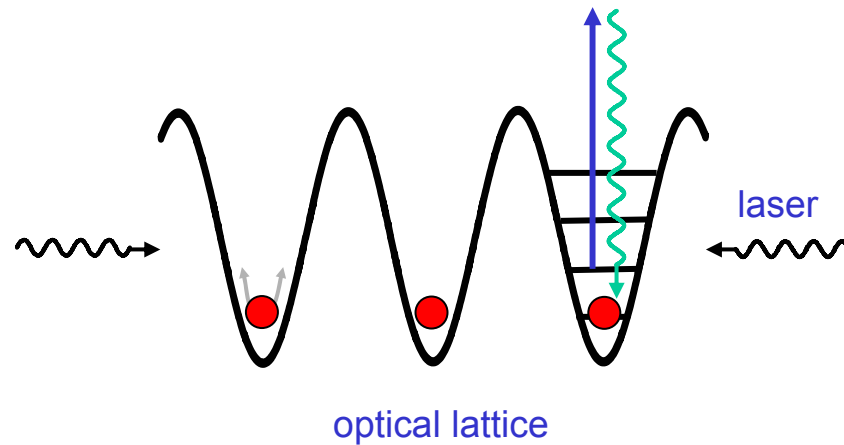
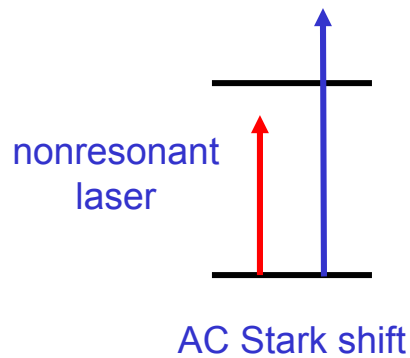
# Remarks: Traps ...

- far-offresonance optical lattice



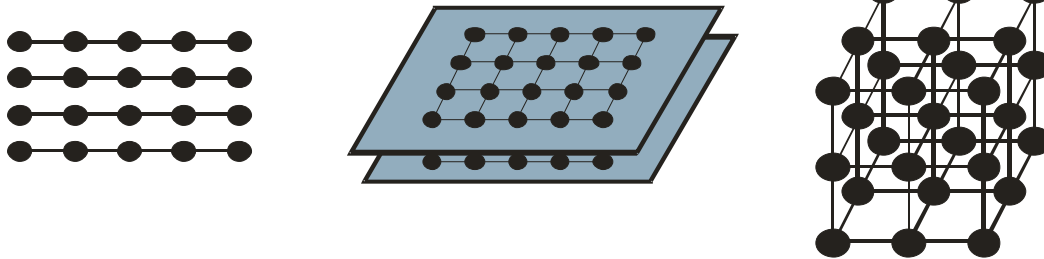
issues:

- ✓ conservative potential
- ✓ single atom loading (?!)
- ✓ laser cooling



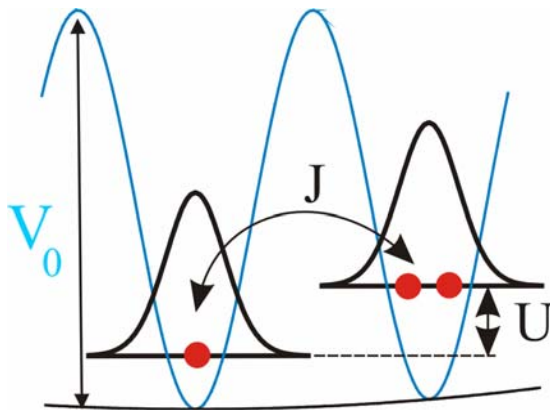
# Remarks: Traps ...

- loading a lattice from a Bose Einstein condensate
  - *regular* filling with exactly 1, 2 or 3 atoms per lattice site via Mott insulator quantum phase transition (Bose Hubbard model)
  - *large* number of atoms



theory: Innsbruck 1998

exp: Munich 2001



Bose Hubbard model

- superfluid phase



- Mott insulator



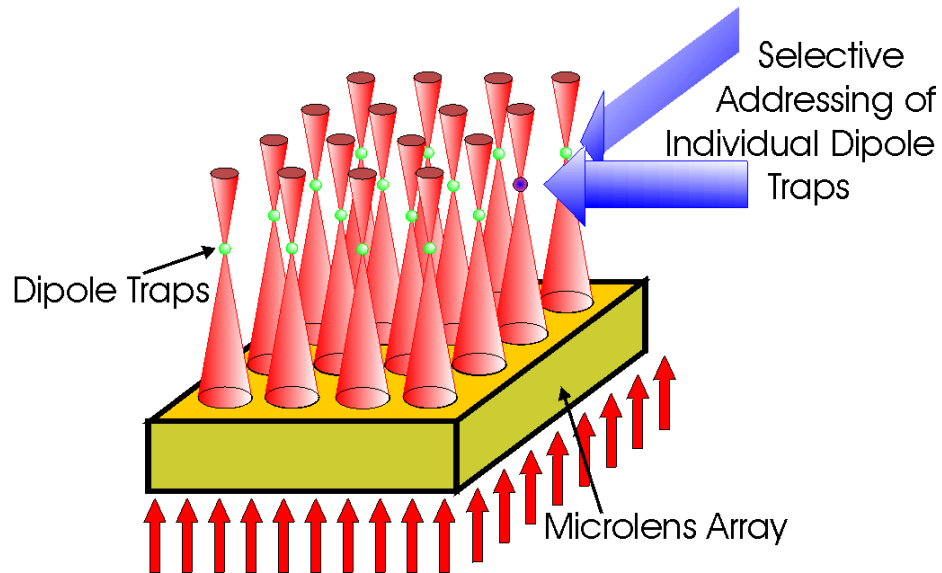
$$b_1^\dagger b_2^\dagger \dots b_M^\dagger |\text{vac}\rangle$$

"Fock states"

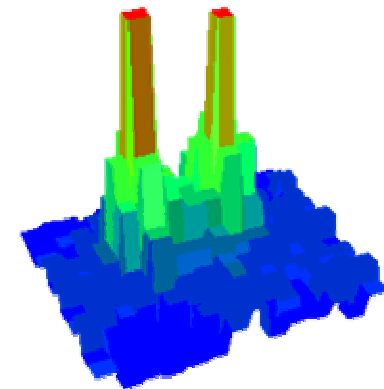
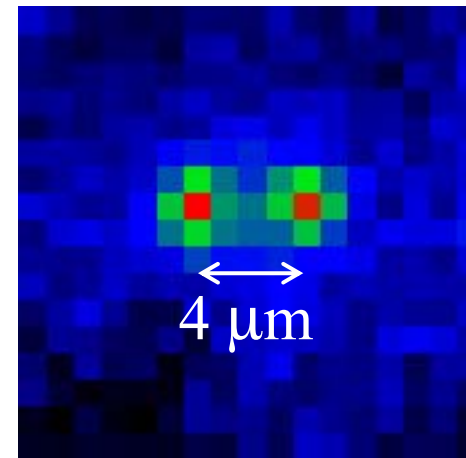
## Remarks: Traps ...

- single atom FORTs

array of FORTs (Hannover)

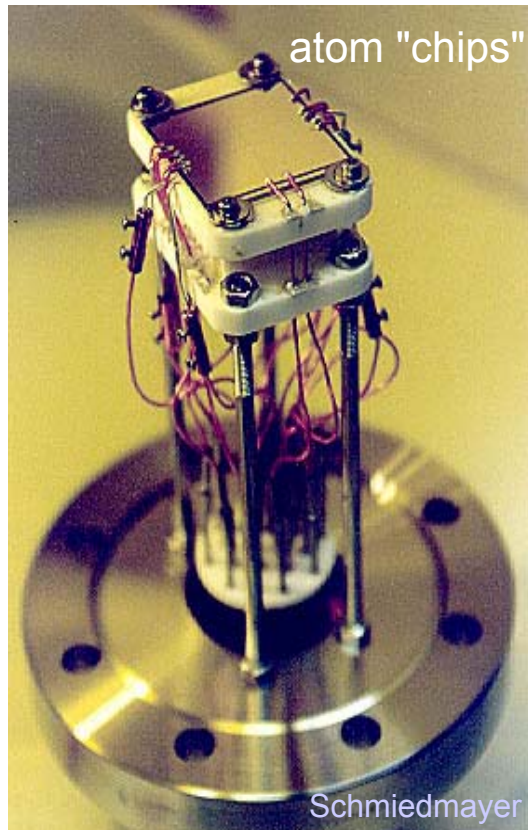


two movable single-atom FORTs (Orsay)



## Remarks: Traps ...

- magnetic traps



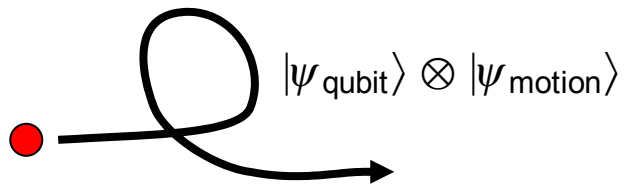
issues:

- ✓ conservative potential  
surface effects (?)
- ✓ single atom loading (?)
- ✓ laser cooling (?)

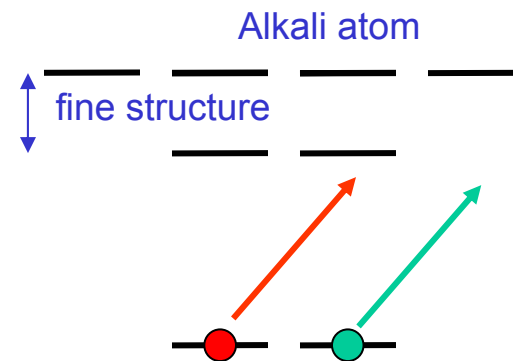
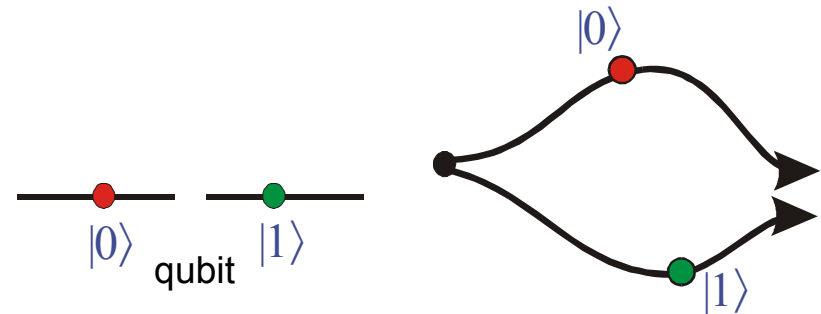
Heidelberg, Munich,  
Harvard, Orsay

## ... and tricks with traps

- We can move atoms around

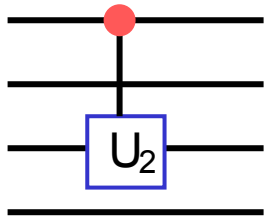


- optical traps: internal state dependent potentials



## 2. Engineering entanglement: two-qubit gates

- implement entanglement of two qubits



example:  
phase gate

$$\begin{aligned} |00\rangle &\rightarrow |00\rangle \\ |01\rangle &\rightarrow |01\rangle \\ |10\rangle &\rightarrow |10\rangle \\ |11\rangle &\rightarrow e^{i\phi}|11\rangle \end{aligned}$$

- How?

- ✓ auxiliary collective mode as data bus

- ✓ controllable two body interactions

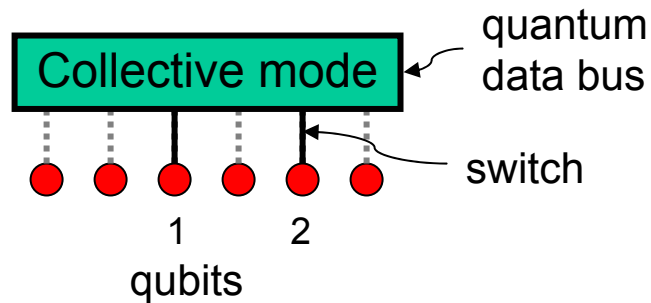
difficult

- ✓ dynamical phases

- ✓ geometric phases

# Concept 1: two-qubit gates via quantum databus

- Entanglement via collective auxiliary quantum degree of freedom



Examples:

Ion traps

Cavity QED

state vector:

$$|\psi\rangle = \sum_{\{x\}} c_x \underbrace{|x_{N-1}x_{N-2}\dots x_0\rangle}_{\text{quantum register}} \otimes \underbrace{|\text{collective mode}\rangle}_{\text{data bus}}$$

gate:



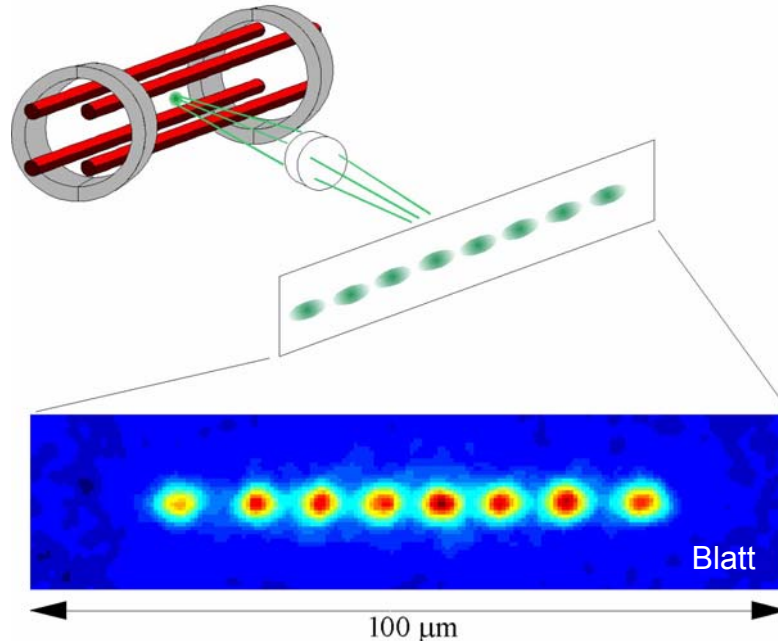
requirement: cooling of the collective mode (= prepare a pure state)

# Ion Trap Quantum Computer

theory: Innsbruck, Aarhus, London, Brisbane ...

exp: NIST Boulder, Innsbruck, Munich, Oxford ..

- Cold ions in a linear trap



- Qubits: internal atomic states
- Quantum gates: entanglement via exchange of phonons of quantized center-of-mass mode

- Achievements:
  - entanglement of four ions
  - Bell measurements
  - individual addressing
  - ground state laser cooling

- State vector

$$|\Psi\rangle = \sum c_x |x_{N-1}, \dots, x_0\rangle_{\text{atom}} |0\rangle_{\text{phonon}}$$

quantum register

database

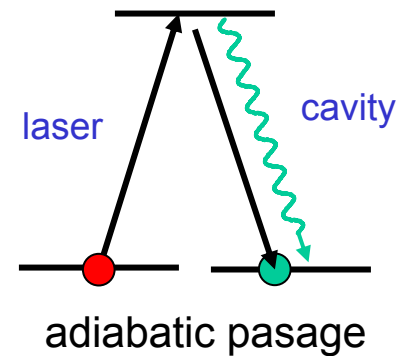
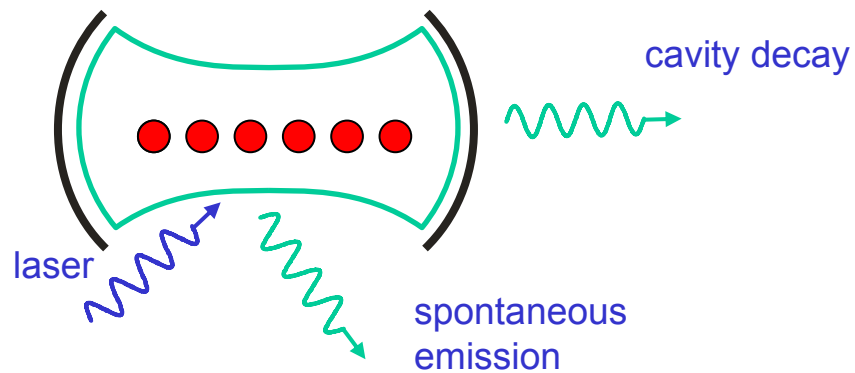


# Optical Cavity QED

theory: Innsbruck, London ...

exp: Caltech, Georgia Tech, Munich

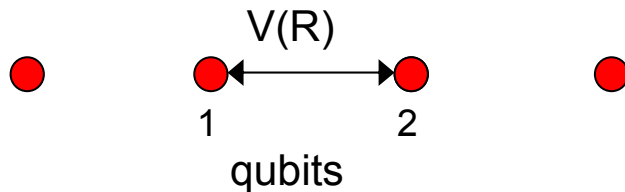
- *optical* photons in a high-Q cavity as "data bus"



decoherence free subspace

## Concept 2: two-qubit gates via two-body interactions

- Controlled two-body interaction



We must design a Hamiltonian

$$H = \Delta E(t) |1\rangle_1 \langle 1| \otimes |1\rangle_2 \langle 1|$$

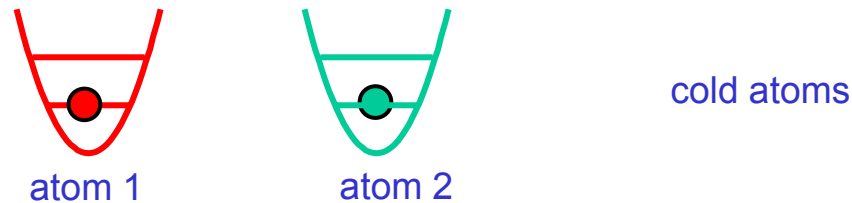
so that

$$|1\rangle_1 \otimes |1\rangle_2 \rightarrow e^{i\phi} |1\rangle_1 \otimes |1\rangle_2$$

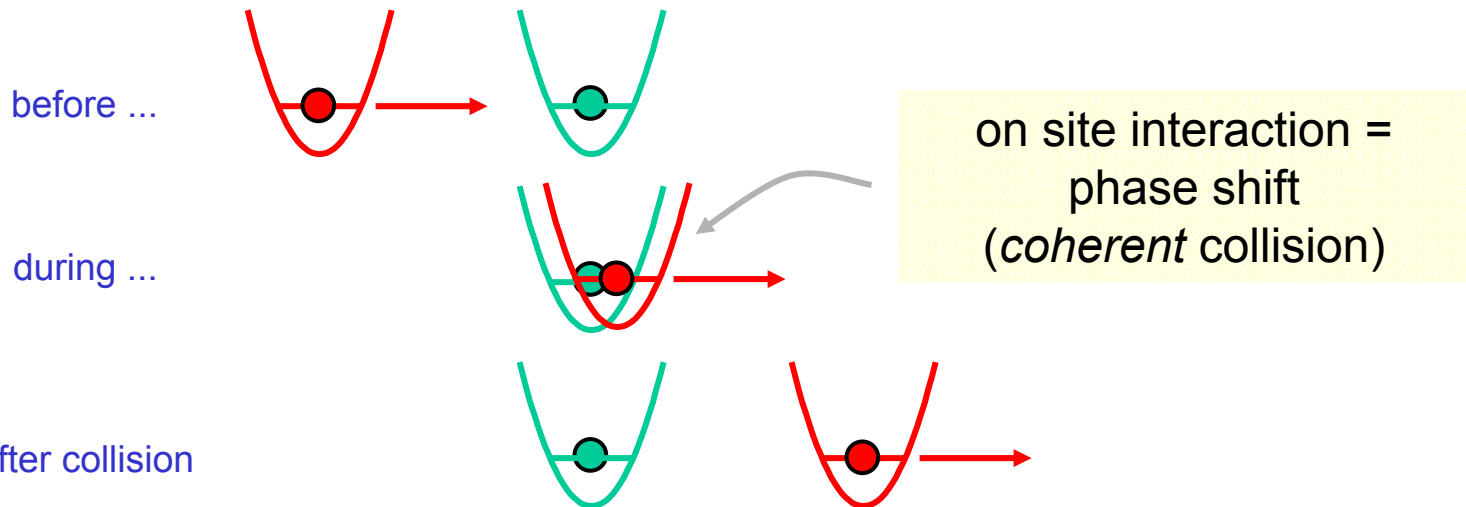
- physical mechanisms
  - ✓ optical dipole – dipole (Albuquerque)
  - ✓ cold collisions (Innsbruck)
  - ✓ Rydberg – Rydberg (Innsbruck + Harvard + Storrs)

# Cold controlled collisions

- consider two atoms in different internal states stored e.g. in an optical lattice



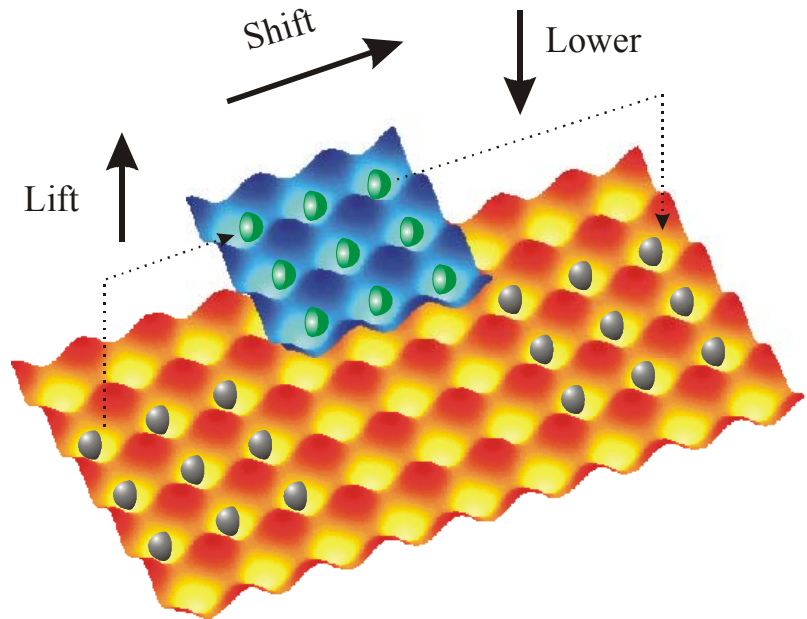
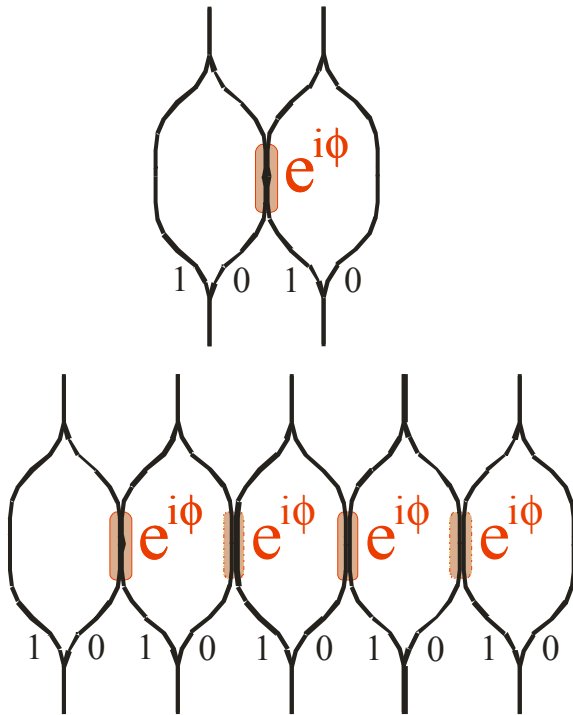
- we *move* the lattice in a state dependent way, and induce a collision: the collision is sufficiently *slow* not to excite oscillations



- phase gate via collisions ...

# Entanglement

- parallelism

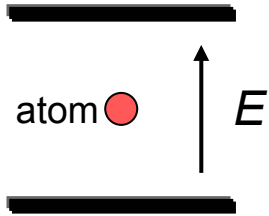


- fidelity  $F \sim 99\%$
- ... slow ☹

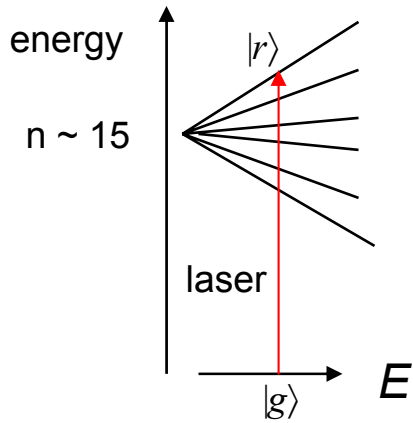
# Rydberg – Rydberg interactions

- Rydberg atom in constant electric field

- setup



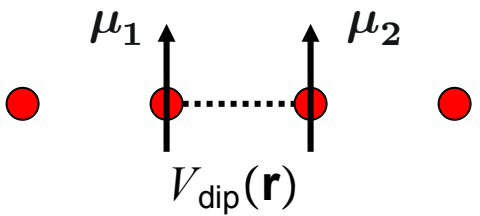
- linear Stark effect



- permanent dipole moment

$\mu \sim n^2$  huge!

- Large dipole-dipole interaction



$E \sim 1\text{ kV/cm}$

$R \sim \lambda_{opt}/2 \sim 300\text{ nm}$

$\Delta E \sim 60\text{ GHz}$  for  $n \sim 15$

$V_{dip} \sim 4\text{ GHz}$

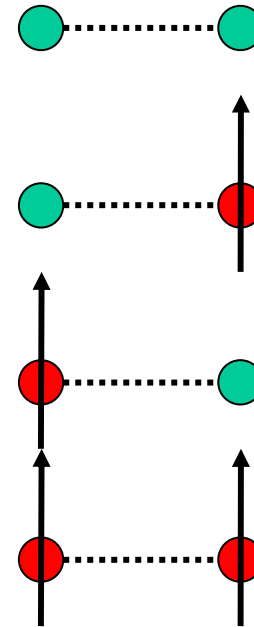
• **2 qubit quantum gate**

$$|00\rangle \rightarrow |00\rangle$$

$$|01\rangle \rightarrow |01\rangle$$

$$|10\rangle \rightarrow |10\rangle$$

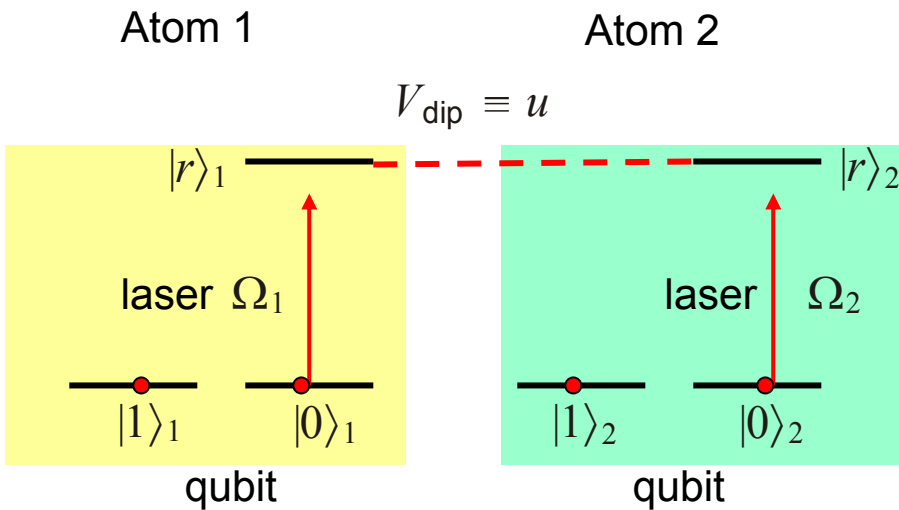
$$|11\rangle \rightarrow e^{i\phi}|11\rangle$$



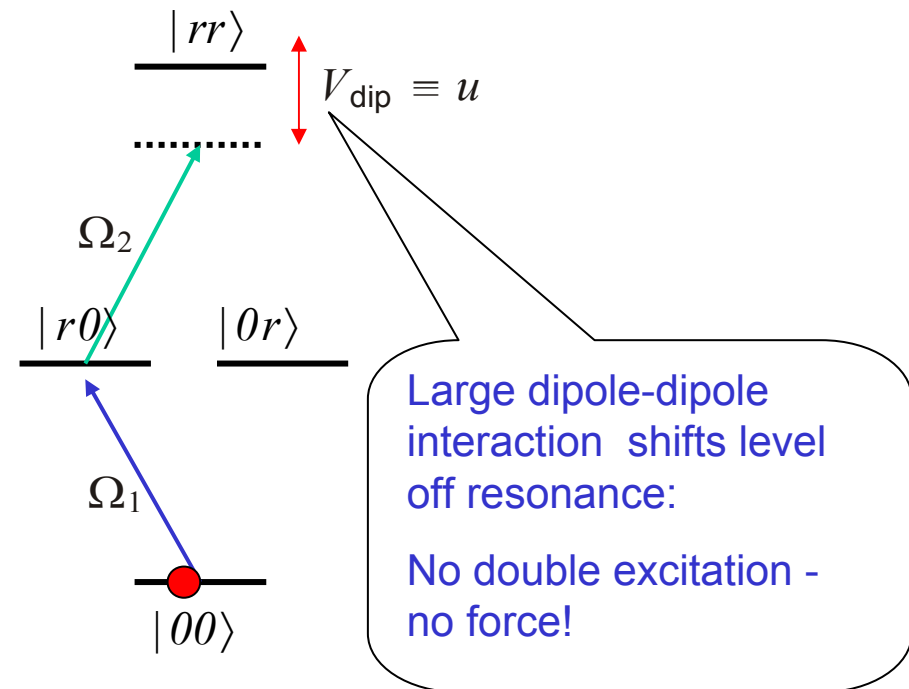
energy shift for time  
interval = phase

# "Dipole blockade"

- atomic configuration



- dipole blockade



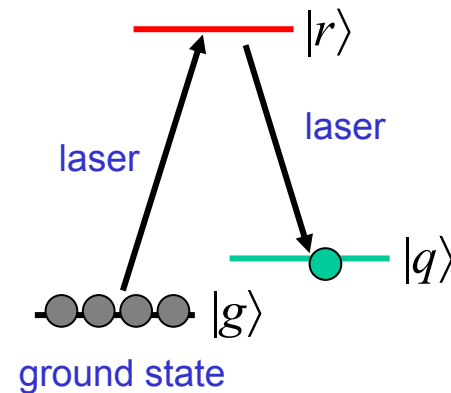
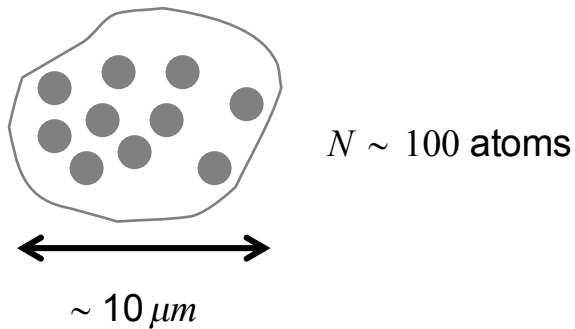
### *3. Mesoscopic atomic ensembles*

- idea
  - dipole blockade mechanism

theory: Harvard + Innsbruck, exp: Storrs, ...

# Configuration

- *mesoscopic* atomic ensembles (instead of microscopic quantum objects)
  - coherent manipulation of *collective excitations* of atomic ensembles



-underlying physics:  
dipole blockade

# Manipulating collective excitations

- ground state

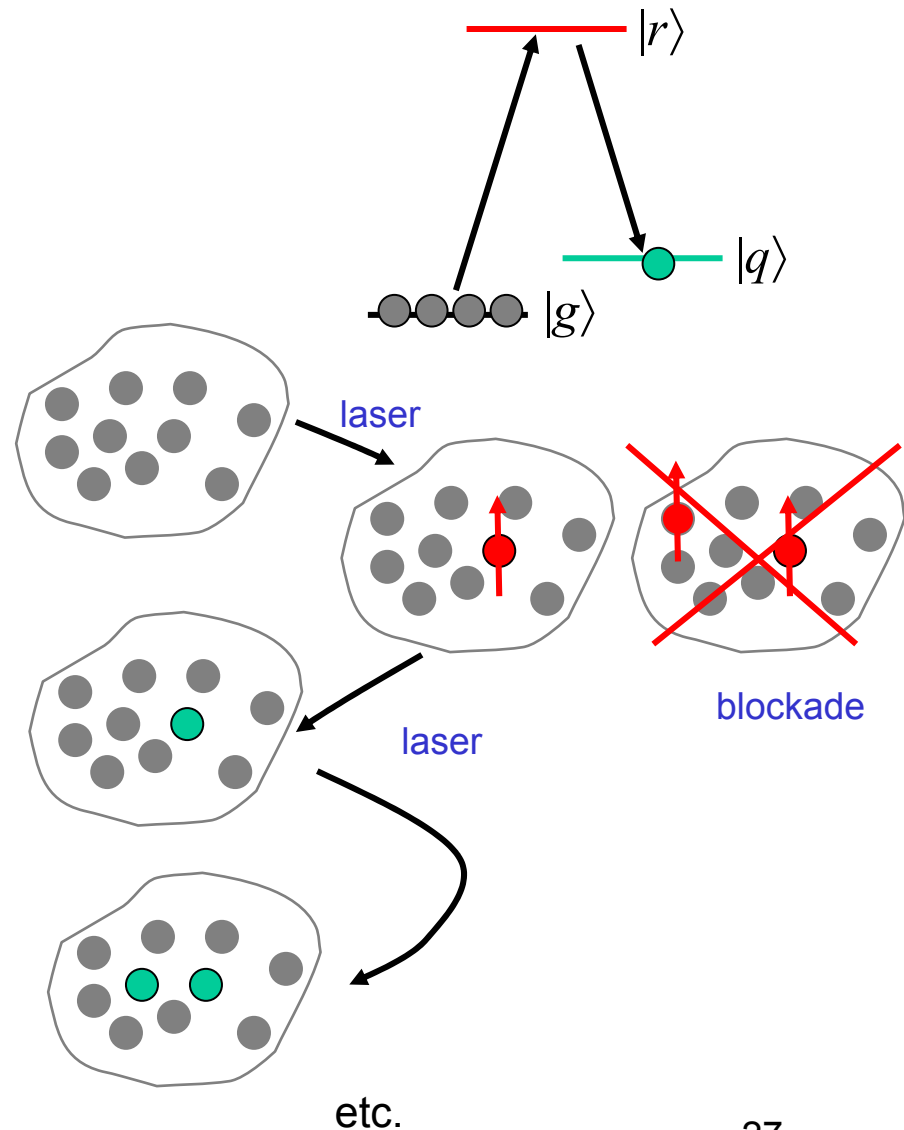
$$|g^N\rangle = |g_1\rangle|g_2\rangle\dots|g_N\rangle$$

- one excitation (Fock state)

$$|g^{N-1}q\rangle \sim \sum_i |g_1\rangle\dots|q_i\rangle\dots|g_N\rangle$$

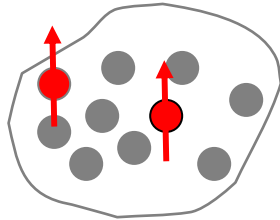
- two excitations

$$|g^{N-2}q\rangle \sim \sum_{i,j} |g_1\rangle\dots|q_i\rangle\dots|q_j\rangle\dots|g_N\rangle$$

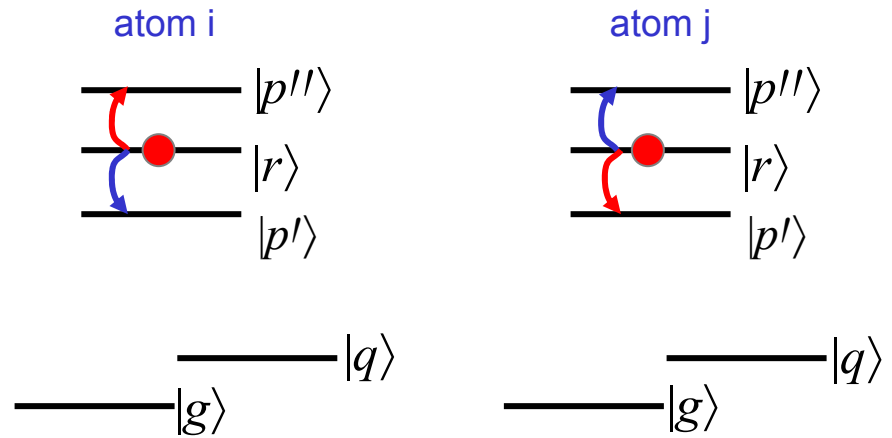


We can store and manipulate qubits.

- actual configuration



hopping via resonant dipole-dipole interaction



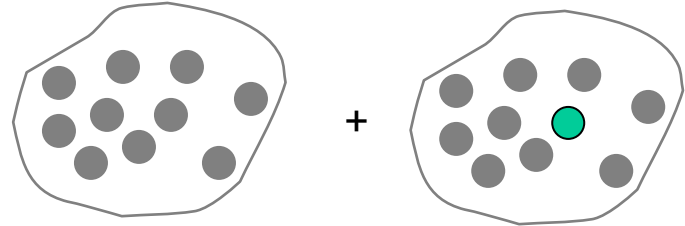
resonance condition:  $E_r - E_{p'} = E_{p''} - E_r$

idea: Cote, Lukin

*cont.*

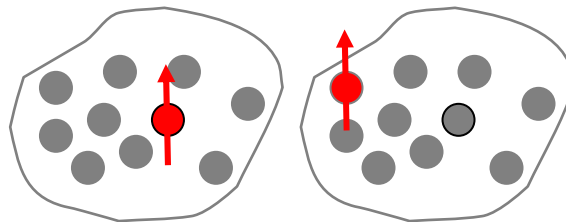
- qubits

$$|\psi\rangle = \alpha|g^N\rangle + \beta|g^{N-1}q\rangle$$



superposition

- entanglement of ensembles

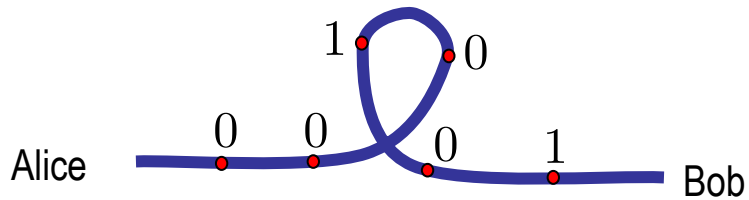


## 4. *Quantum Communication*

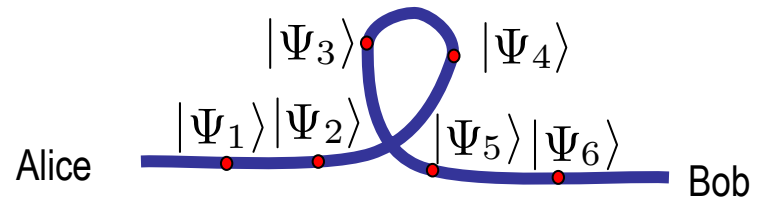
- Question:
  - what can we do with 10 qubit quantum computers?
- Example:
  - quantum communication with memory and quantum error correction = quantum repeater

# Quantum Communications

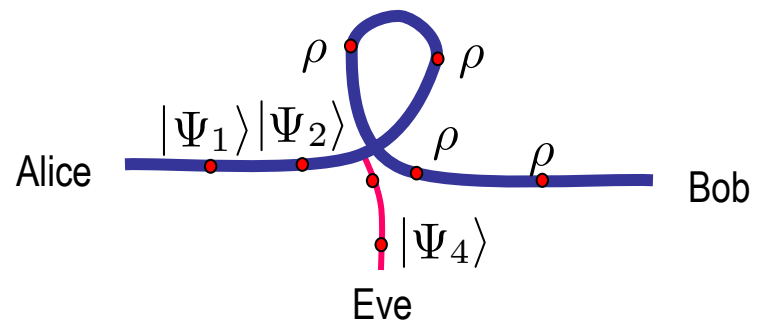
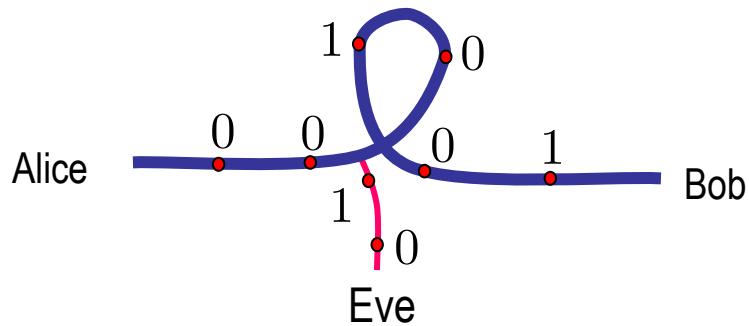
- classical communications



- quantum communications



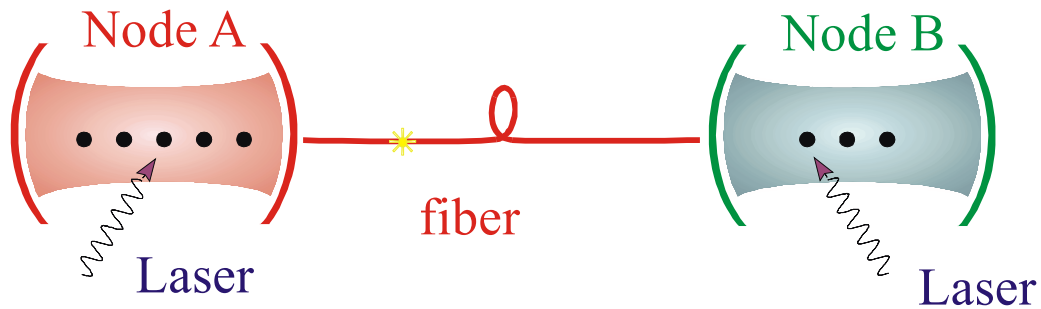
- ✓ networks
- ✓ cryptography



# Optical Interconnects

- A cavity QED implementation

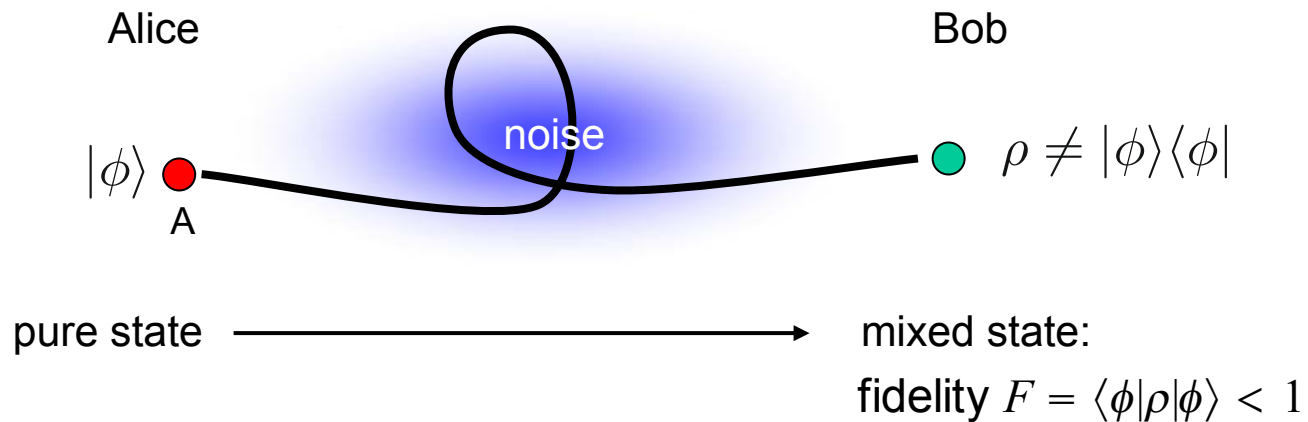
Optical cavities connected by a quantum channel



• memory: atoms → • databus: photons → • memory: atoms

# Quantum Communication Protocols

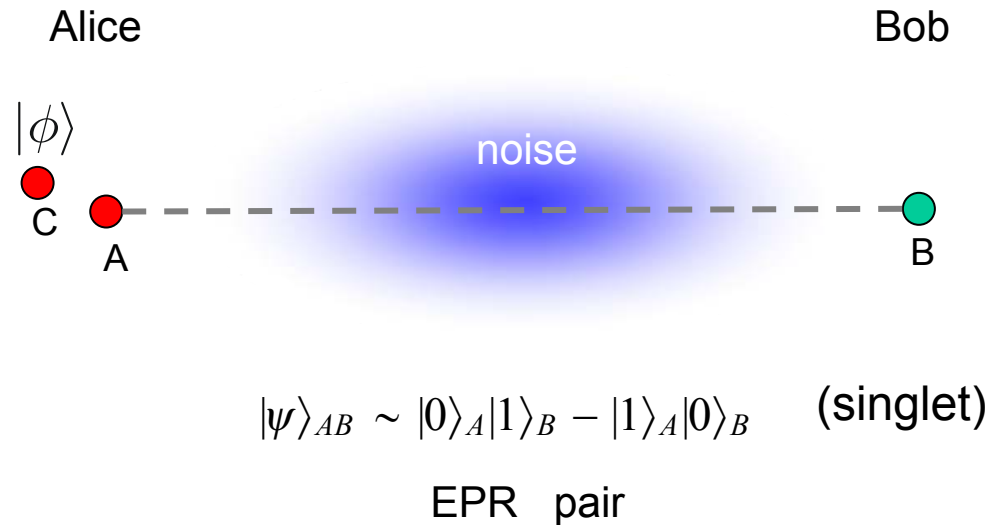
- Problem: the basic problem of quantum communication is noise / decoherence



- Solution: quantum communication in the presence of noise is based on
  - teleportation
  - purification (error correction)

# 1. Teleportation

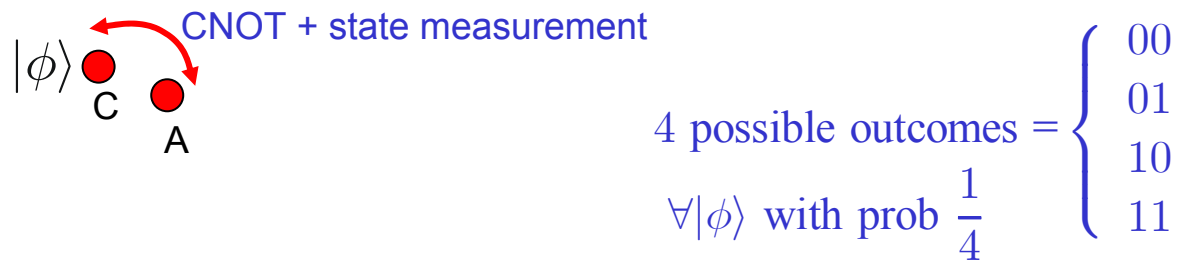
Bennett et al. PRL '93; exp with photons: Innsbruck, Rome, Caltech



- Assume: Alice and Bob share a singlet state (given resource)
- Idea: Alice can transmit a qubit to Bob without physically sending the qubit

## Teleportation protocol:

1. Alice performs a CNOT followed by a measurement of A and C



2. Alice tells Bob the measurement outcome



3. Rotation



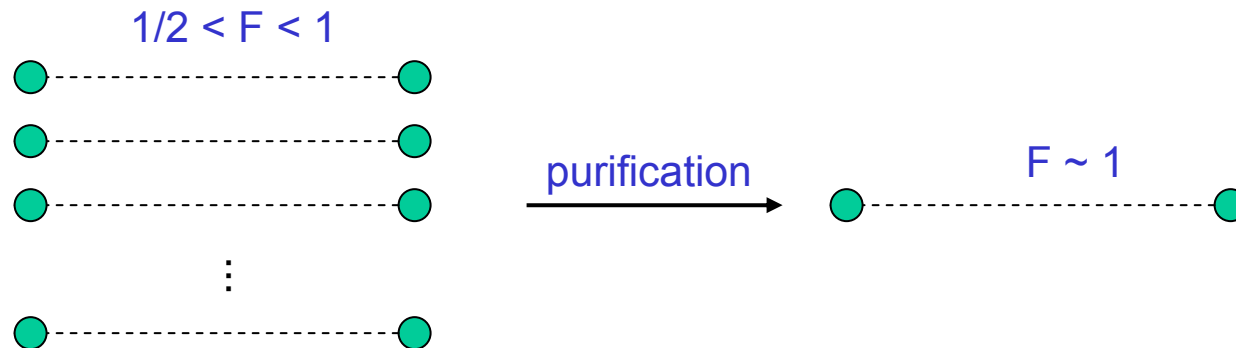
The state has been teleported to B.

How do we get the EPR pair?

## 2. Purifying EPR pairs

Bennett et al., Deutsch et al. PRL '95

- A noisy quantum channel allows us to generate *many noisy* EPR pairs.



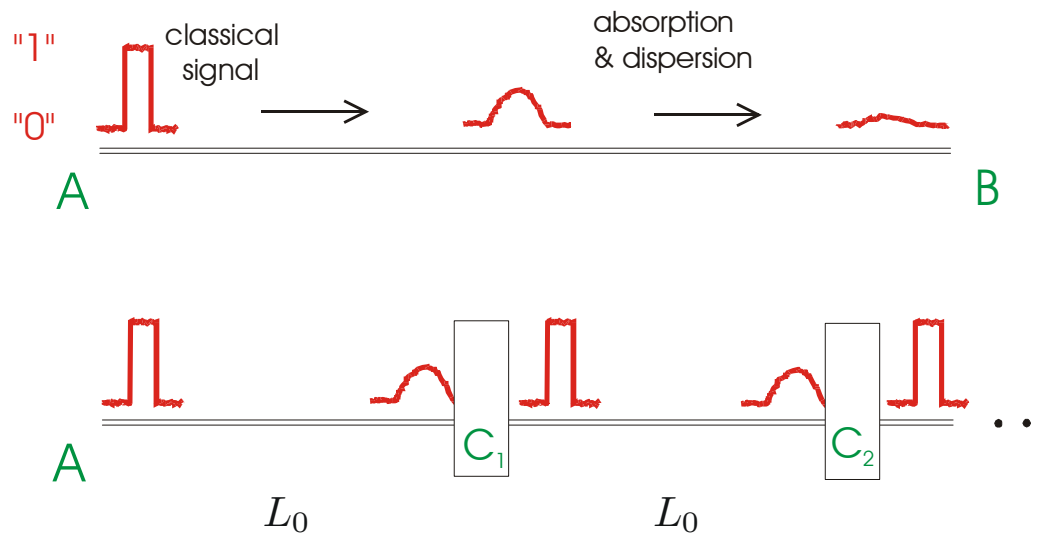
If  $F > 1/2$  we can purify and obtain one EPR pair with  $F \sim 1$ .

To do this we need a small quantum computer.

- In summary: quantum communication via a noisy channel:
  - we generate one high fidelity EPR pair by purification
  - we teleport the quantum state

### 3. Quantum Repeater

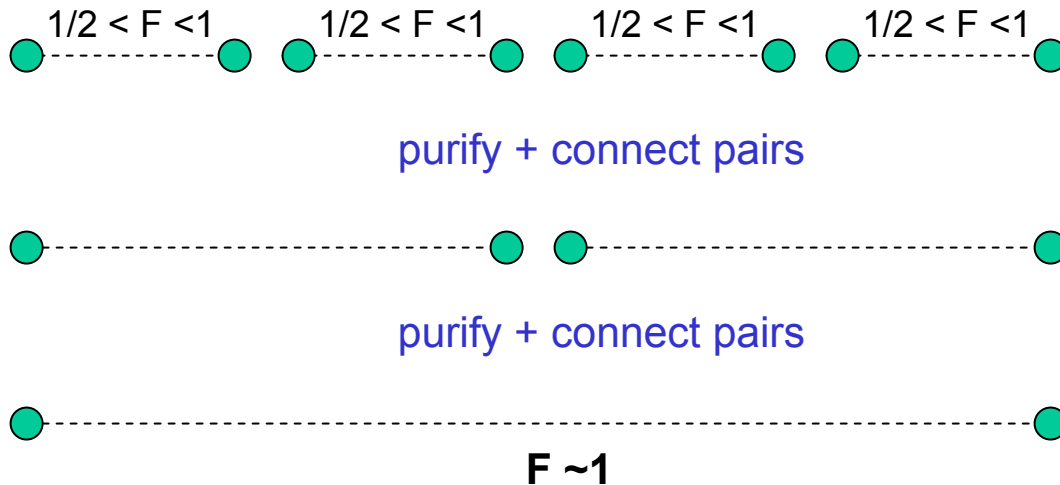
- classical repeater



- quantum repeater:
  - we cannot clone a quantum state!?
  - Idea: purify segments of EPR pairs ... and teleport H. Briegel et al. PRL '98

cont.

- Quantum repeater protocol: generate a long distance EPR pair

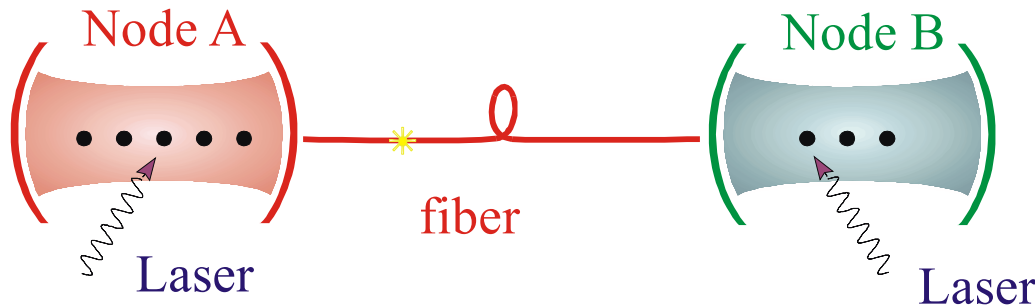


We use this EPR pair to teleport the quantum state.

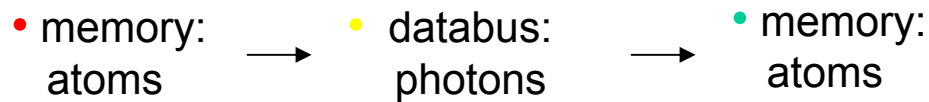
- required resources for a photonic channel
  - ~ 10 qubit quantum computer for 10.000 km
  - tolerate errors on the few % level

## Other (easier?) implementations

- single atoms, single photons and high-Q cavities: "quantum engineering"

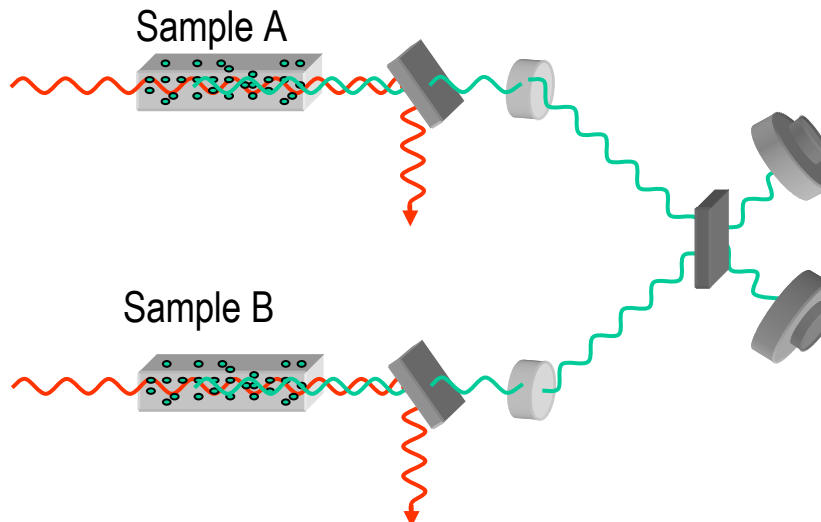


- ✓ deterministic
- ✓ strong coupling on single quantum level



difficult !

- atomic ensembles, free space, or low-Q cavities: "quantum gambling"

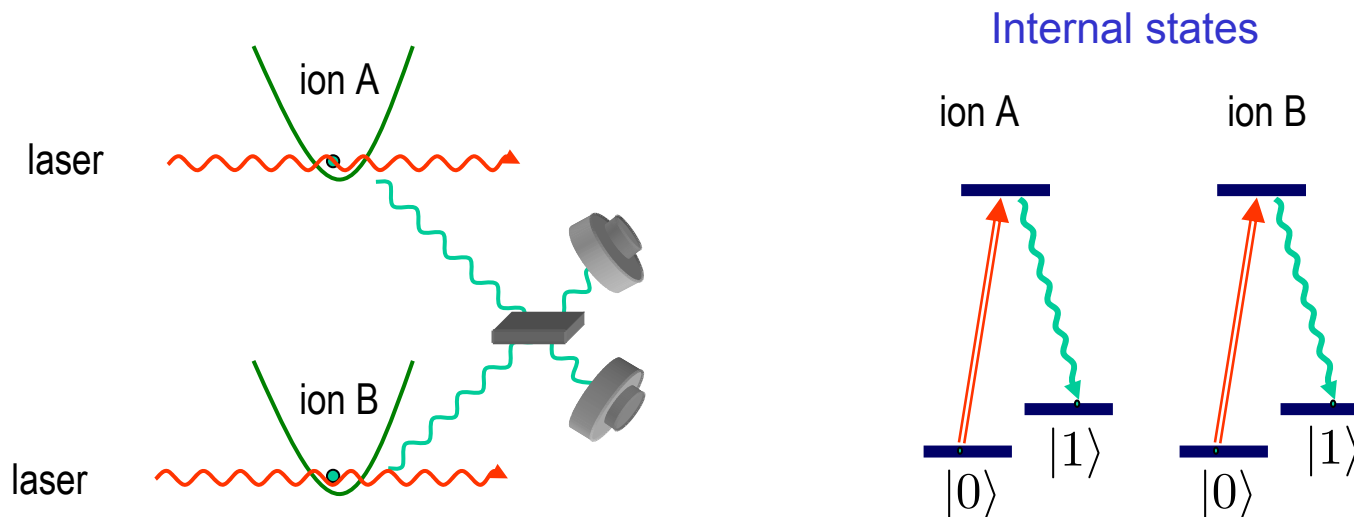


- ✓ probabilistic
- ✓ scheme
- ✓ noise tolerant
- ✓ linear optics

simpler !

To illustrate the ideas ...

## Single Atoms

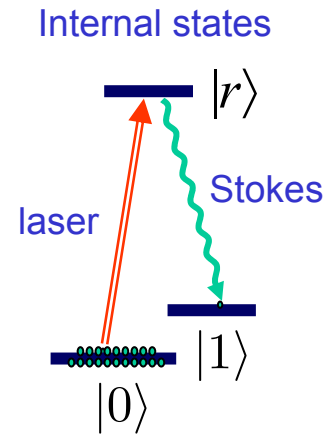
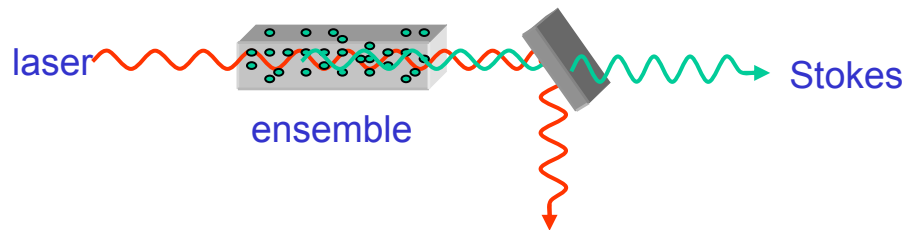


- Weak (short) laser pulse, so that the excitation probability is small.
- If no detection, pump back and start again.
- If detection, an entangled state is created because we do not know which atom emitted the photon

$$|0, 1\rangle \pm |1, 0\rangle$$

# Atomic Ensembles

- system: cloud of cold atoms



- Raman process:

$$|0\rangle^{\otimes N} \equiv |\text{vac}\rangle \text{ (atomic ground state)}$$

$$\rightarrow \frac{1}{\sqrt{N_a}} \sum_{i=1}^{N_a} |0_1 \dots 1_i \dots 0_{N_a}\rangle \equiv a^\dagger |\text{vac}\rangle \text{ (single atomic excitation)}$$

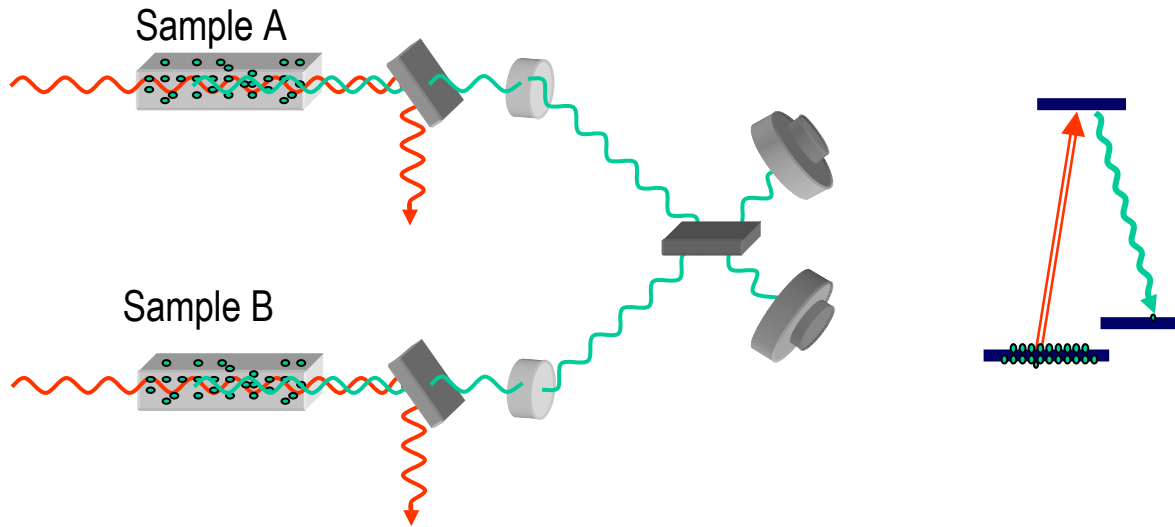
$$[a, a^\dagger] \approx 1 \text{ (for weak excitation)}$$

- state of atomic collective mode + Stokes photon

$$|\phi\rangle = |\text{vac}\rangle + \sqrt{p_c} a^\dagger c_{\text{Stokes}}^\dagger |\text{vac}\rangle + O(\sqrt{p_c}^2) \quad (p_c \ll 1)$$

... analogous to parametric downconversion

## Generation of Entanglement (Ensembles)



$$|\phi\rangle_A \otimes |\phi\rangle_B = (|\text{vac}\rangle_A + \sqrt{p_c} a^\dagger c_s^\dagger |\text{vac}\rangle_A) \otimes (|\text{vac}\rangle_B + \sqrt{p_c} b^\dagger c_s^\dagger |\text{vac}\rangle_B)$$

measurement gives

$$\begin{aligned} |\psi_{AB}^\pm\rangle &= (a^\dagger \pm b^\dagger) |\text{vac}\rangle \\ &\equiv |1_a, 0_b\rangle \pm |0_a, 1_b\rangle \end{aligned}$$

We have generated entanglement between collective atomic states

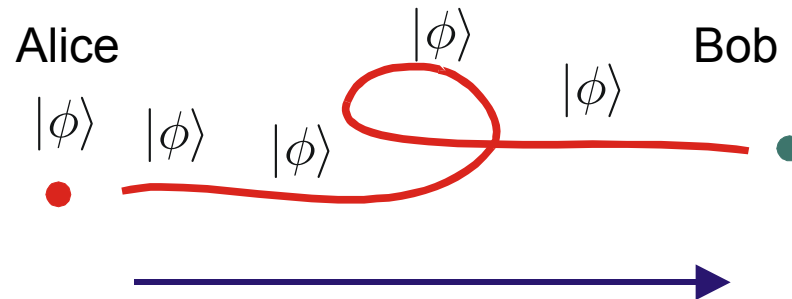
# Continuous Variable Teleportation

- Instead of qubits we consider now continuous variable quantum states

$$|\phi\rangle = \int dx |x\rangle \phi(x) \quad \hat{x} \dots \text{position} \quad [\hat{x}, \hat{p}] = i$$

$$\hat{p} \dots \text{momentum}$$

- transmission of a cv state



- continuous variable teleportation



Vaidman  
Braunstein  
Kimble (exp)

$$|\text{EPR}\rangle_{AB} \sim \int dx |x\rangle_A |x\rangle_B$$

$$\sim \int dp |p\rangle_A | -p\rangle_B$$

$$(\hat{x}_A - \hat{x}_B)|\text{EPR}\rangle = x_1|\text{EPR}\rangle$$

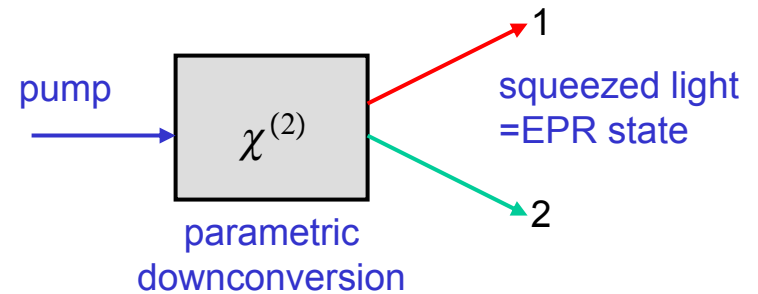
$$(\hat{p}_A + \hat{p}_B)|\text{EPR}\rangle = p_1|\text{EPR}\rangle$$

# Teleportation with Squeezed Light

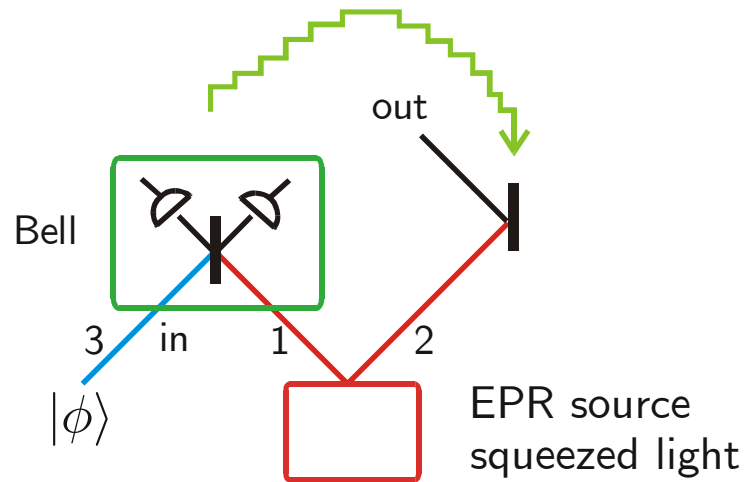
S. Braunstein, H.J. Kimble et al., PRL '98; Science '99

- Two-mode squeezed light:

electric field  $E^{(+)} \sim a e^{ikx - i\omega t}$   
 $\searrow$   
 $= \hat{x} + i\hat{p}$   
 quadrature components

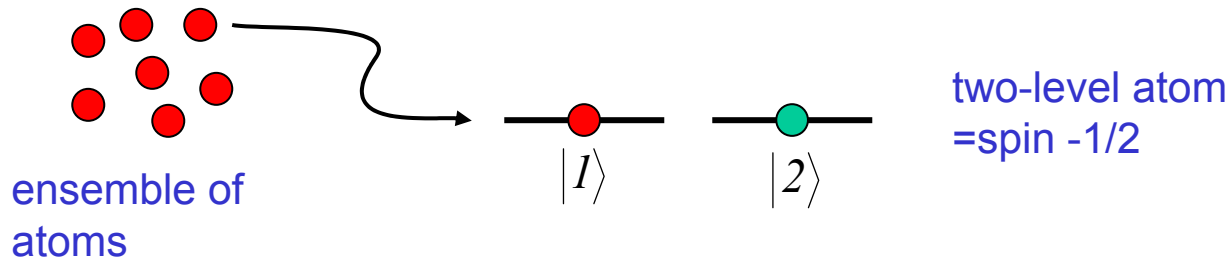


- Scheme



# Atomic ensembles as quantum memory for cont var states

- We consider an ensemble of N atoms

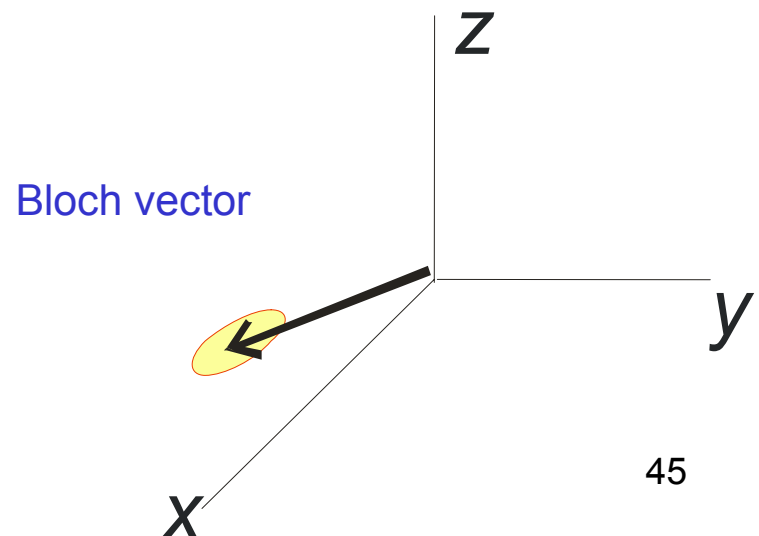


- a collection of two-level atoms can be described in terms of a collective "angular momentum"

$$\vec{S}^a = \sum_{\mu=1}^N \frac{1}{2} \vec{\sigma}^{(\mu)}$$

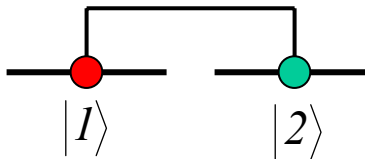
collective angular momentum

two-level atom = spin -1/2



## atoms cont.

- superposition of the two ground states: **coherent spin state**



$$\left[ \frac{1}{\sqrt{2}} (|1\rangle + |2\rangle) \right]^{\otimes N}$$

**Bloch vector**

$$\langle \vec{S}^a \rangle = (\langle S_x^a \rangle, \langle S_y^a \rangle, \langle S_z^a \rangle) = \left( \frac{N_a}{2}, 0, 0 \right)$$

- quantum fluctuations

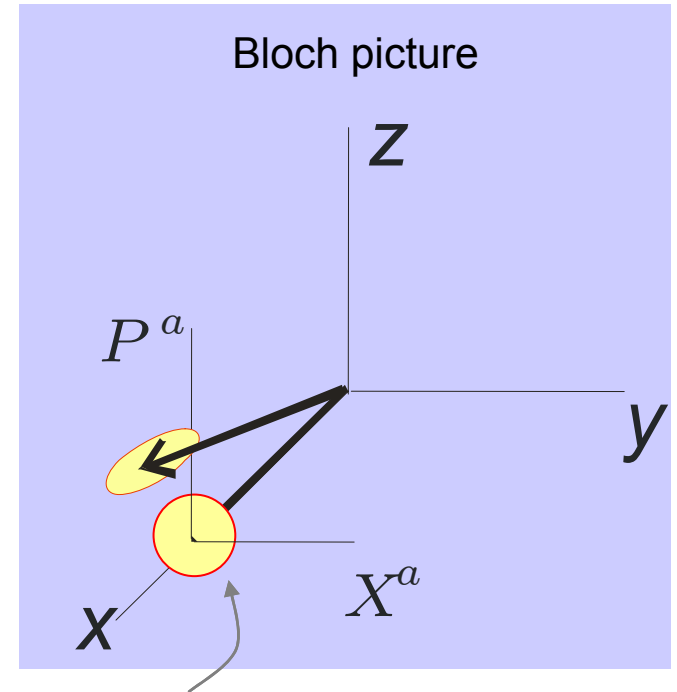
$$[S_y^a, S_z^a] = iS_x^a \quad \Delta S_y^a \Delta S_z^a \geq \frac{1}{2} |\langle S_x^a \rangle|$$

we treat  $S_x^a$  classically and rescale

$$[X^a, P^a] = i$$

$$\Delta X^a \Delta P^a \geq \frac{1}{2}$$

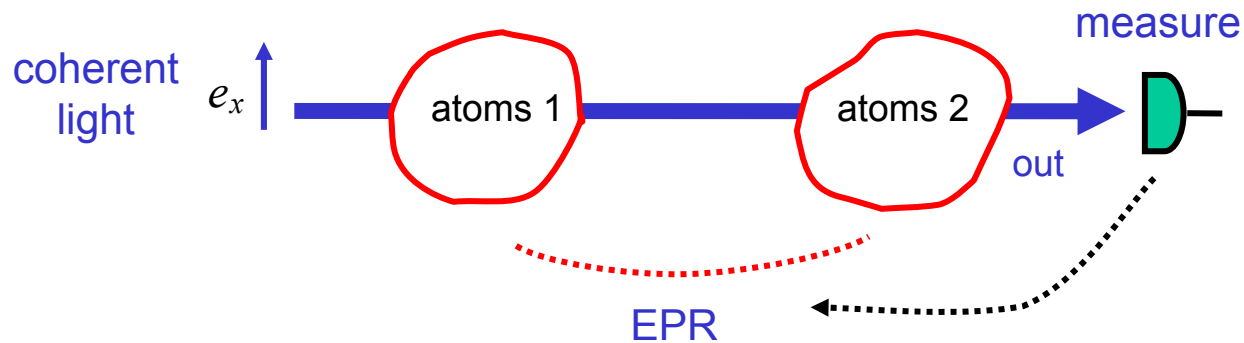
canonical commutation relations



- ✓ coherent spin state = vacuum state
- ✓ there are *many* cv quantum states around it:

$$|\psi^a\rangle = \int dX^a |X^a\rangle \psi(X^a)$$

# Teleportation with coherent light + atomic ensembles



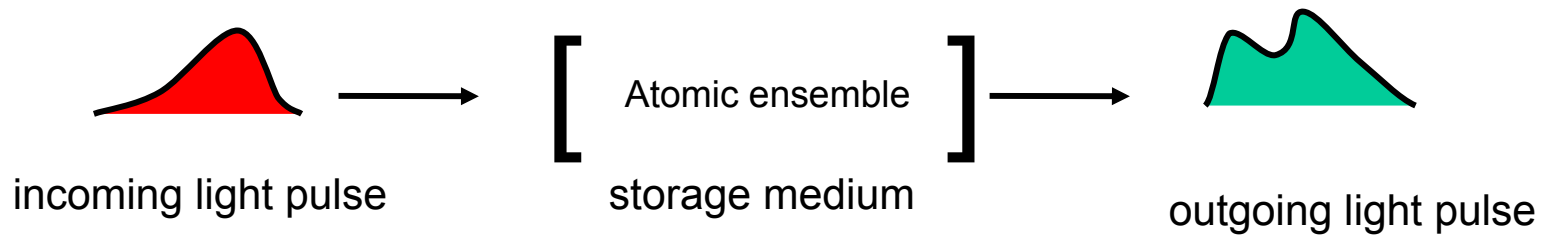
*measurement* projects atomic ensembles  
into continuous variable EPR state

$$|\text{EPR}\rangle \sim \int dP |P\rangle_A | -P\rangle_B = \int dX |X\rangle_A |X\rangle_B$$

- ✓ theory: Innsbruck
- ✓ experiment: E. Polzik et al. (Aarhus), Nature 2001

# Atomic ensembles: quantum memory for light

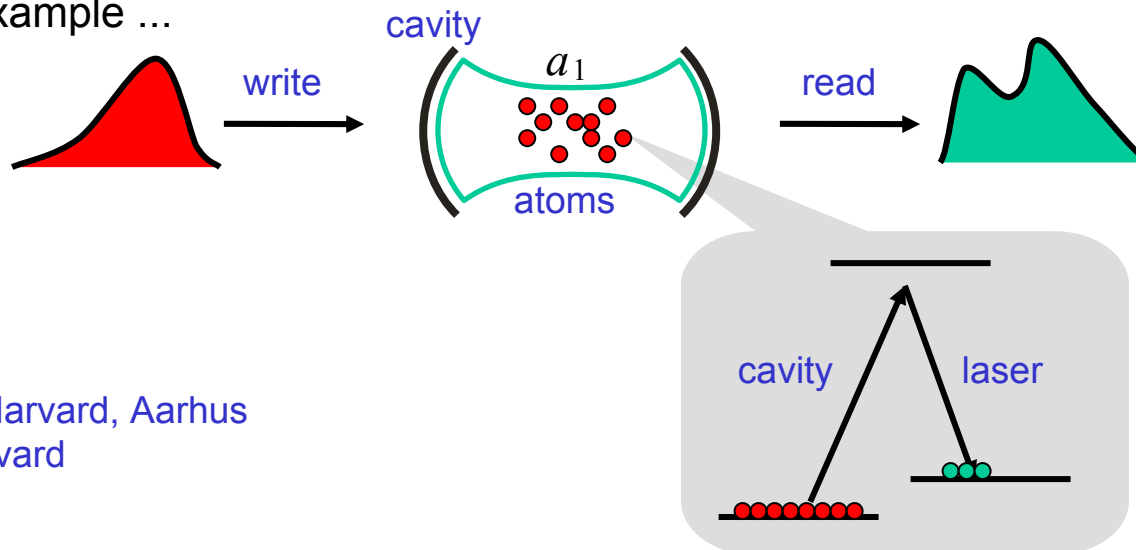
- purpose



- ✓ unknown (arbitrary) state
- ✓ known shape of wave packet

- ✓ same state
- ✓ reshaping

- how? example ...



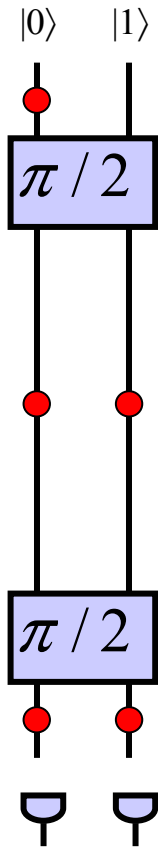
## 4. *Precision measurements*

- maximally entangled states and squeezed atomic states for precision measurement beyond the standard quantum limit

theory: NIST Boulder, Innsbruck, Aarhus, Georgia Tech, Harvard

exp: NIST Boulder, Aarhus, Rochester

# Ramsey method



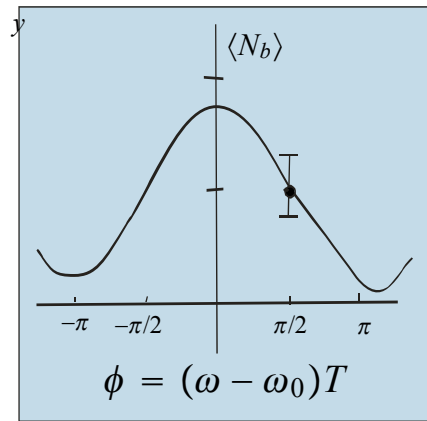
$$|0\rangle \rightarrow \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$\rightarrow \frac{1}{\sqrt{2}}(|0\rangle + e^{-i(\omega_L - \omega_0)t}|1\rangle)$$

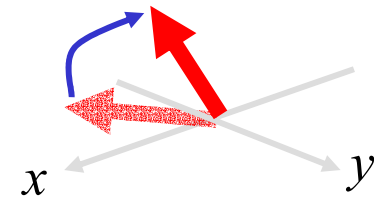
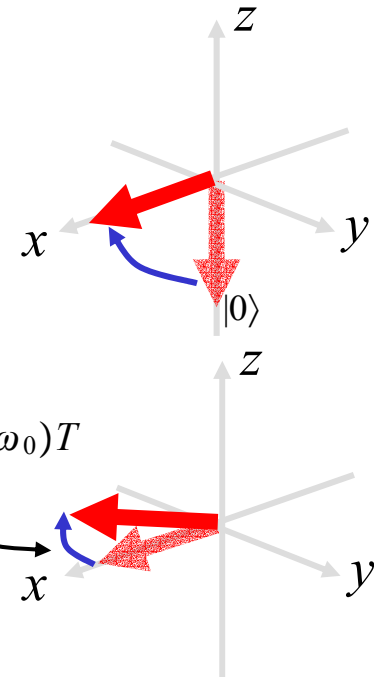
$$\cos\left(\frac{1}{2}(\omega_L - \omega_0)t\right)|0\rangle + \sin\left(\frac{1}{2}(\omega_L - \omega_0)t\right)$$

# of atoms in  $|0\rangle$

$$P_0(T) = \cos^2\left(\frac{1}{2}(\omega_L - \omega_0)T\right)$$



Bloch picture



We compare ...

- N independent atoms



$$\Delta\omega_{\text{SQL}} = \frac{1}{T\sqrt{n_{\text{rep}}}} \frac{1}{\sqrt{N}}$$

standard quantum noise limit

Remarks:

- ✓ N is limited: density / collisions
- ✓ T is limited: decoherence

- N entangled atoms



$$\Delta\omega_{\text{ent}} = \frac{1}{T\sqrt{n_{\text{rep}}}} \frac{1}{f(N)} \geq \frac{1}{T\sqrt{n_{\text{rep}}}} \frac{1}{N}$$

Heisenberg limit:  
maximally entangled state

$$|0000\rangle + |1111\rangle$$

- figure of merit: to achieve the same uncertainty ...

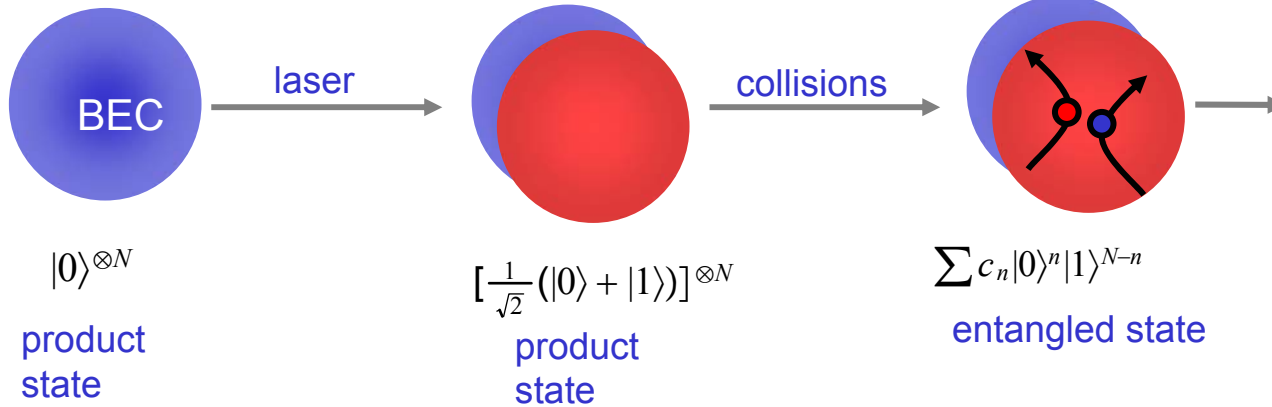
$$\xi^2 = \frac{(n_{\text{rep}})_{\text{SQL}}}{(n_{\text{rep}})_{\text{ent}}} = \frac{N}{f(N)^2} \geq \frac{1}{N}$$

We want  $\xi^2 \ll 1$

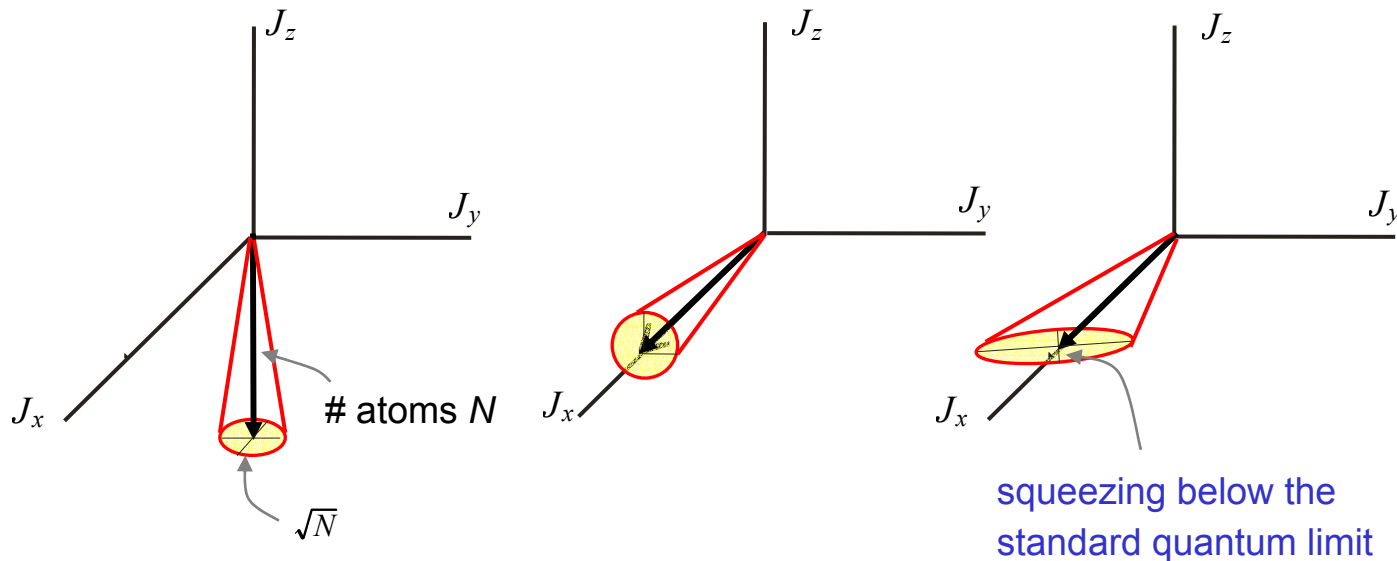
- present experiments: NIST Boulder 4 ions entangled

## Example: Spin Squeezing with BEC

- Entanglement via collisions



- Bloch vector picture



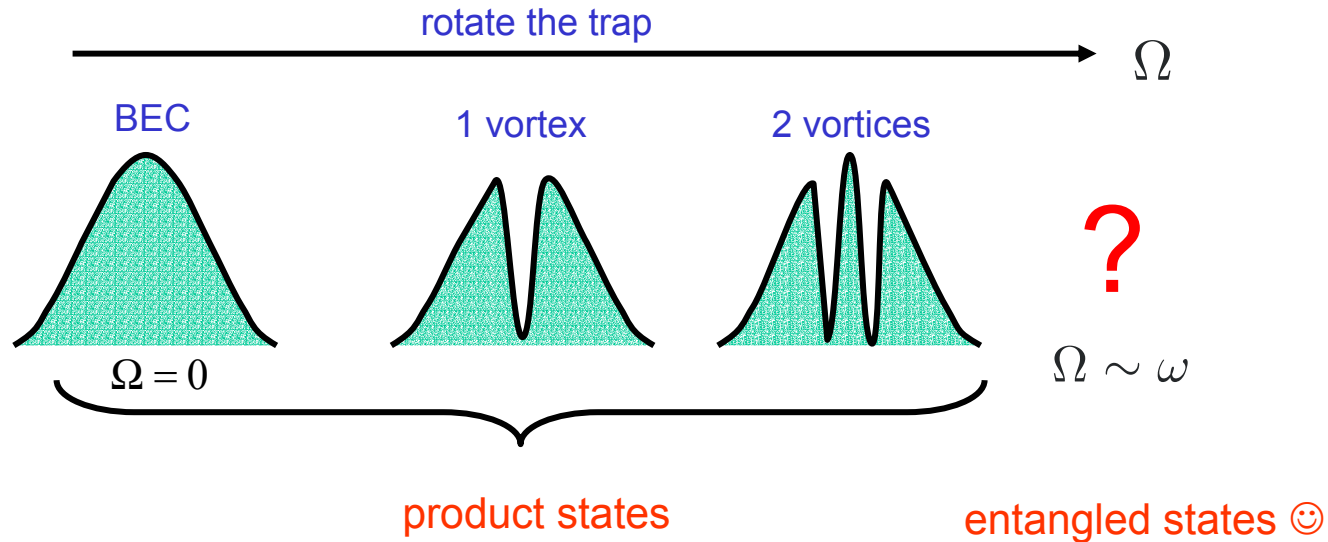
## 5. Topological quantum computing

concept: Kitaev, Preskill, Freedman

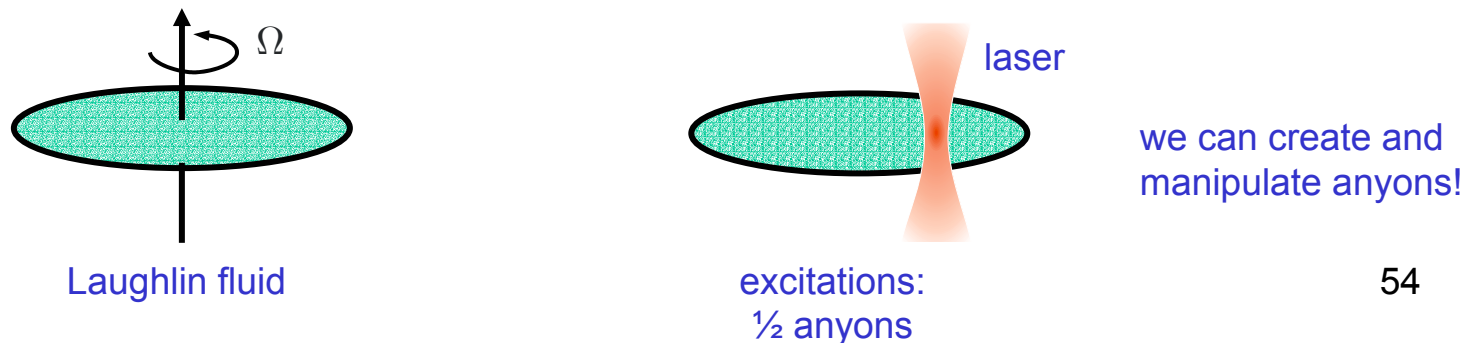
- engineering model systems with *topologically protected ground states*: indistinguishable under local operations
- Fractional quantum Hall effect: electron in 2D + magnetic field: isolated ground state with gapped excitations
- on a closed surface of genus  $g$ : renders the *ground state degenerate*.
- transformation of degenerate ground states into each other via motion of flux quantum around nontrivial topological paths
- Rem.: so far no atomic physics models

# A (small) first step: $\frac{1}{2}$ - (abelian) anyons in small BECs

- quantum degenerate Bose gas: 2D and rotating trap

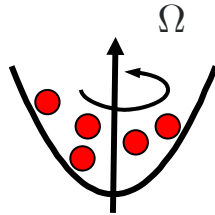


- what happens is a fractional quantum Hall scenario with bosons



# 1. System: $N$ bosonic atoms

- $N$  bosons



- ✓ harmonic trapping potential  $\omega$
- ✓ rotation  $\Omega$

- Hamiltonian in rotating frame

$$H = \frac{1}{2} \sum_{i=1}^N \left( -\nabla_i^2 + r_i^2 - 2\frac{\Omega}{\omega} L_{iz} \right) + \eta \sum_{i<j}^N \delta(\mathbf{r}_i - \mathbf{r}_j)$$

length scale:

motion in trap

rotation

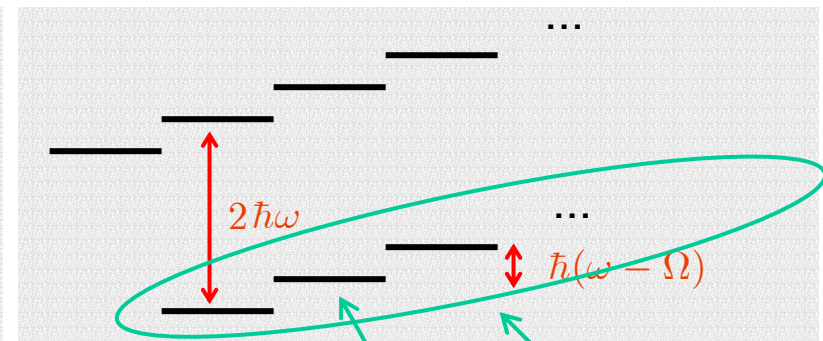
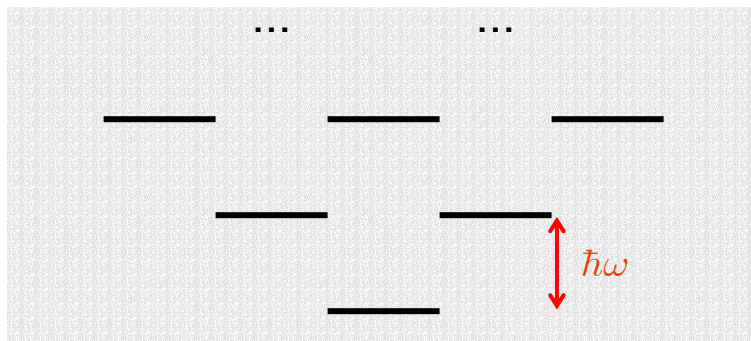
contact interaction

$$\ell = (\hbar/m\omega)^{1/2}$$

- In the limit  $\Omega \sim \omega$  equivalent to (fermionic) quantum Hall Hamiltonian  
Gunn & Wilkin '00

## 2. Energy spectrum

- independent atoms



Landau wave functions:

$$z^m e^{-\frac{1}{2}|z|^2}$$

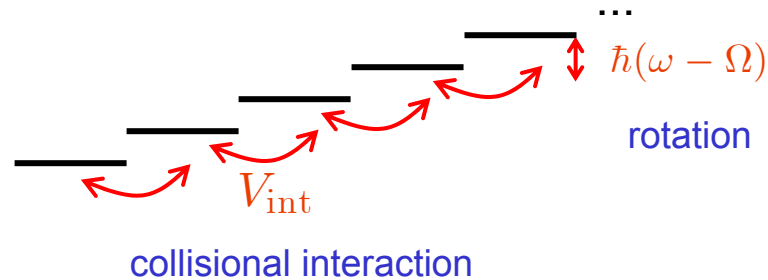
$\curvearrowright x + iy$

"Landau levels":

$\sim$  degenerate

- we assume: dynamics restricted to first Landau levels  $kT \ll \hbar\omega$

- interacting atoms
  - dynamics in first "Landau level"



- for  $\Omega \sim \omega$  a parameter regime of *effectively strong* interactions

$$V_{\text{int}} \gg \hbar(\Omega - \omega)$$



external parameter

✓ strongly correlated qu fluid

✓ dilute gas!

cont.

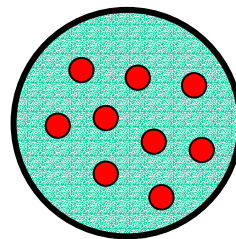
- ground state:  $\frac{1}{2}$  - Laughlin state

physics: atoms want to be in a *collisional dark state* to minimize energy:

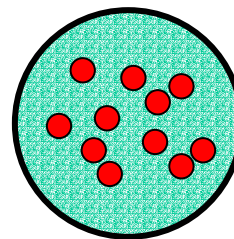
$$V_{\text{int}}\psi = 0$$

$$\psi[z] = \prod_{i < j} (z_i - z_j)^2 \prod_k e^{-|z_k|^2/2}$$

built from lowest Landau levels      avoid interaction      bosons



Laughlin

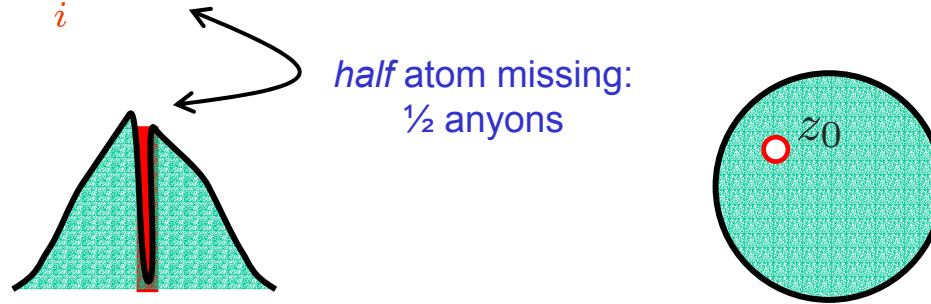


independent  
atoms

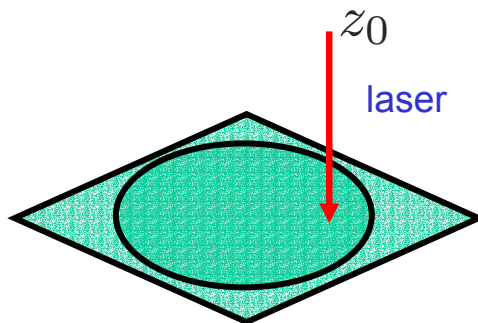
### 3. Excitations: $\frac{1}{2}$ anyons

- quasihole at position  $z_0$

$$\psi_{z_0}[z] = \prod_i (z_i - z_0) \psi[z]$$



- creating a quasiparticle: piercing with an offresonant laser



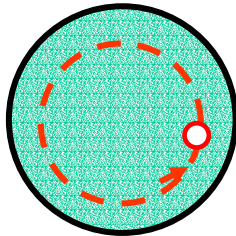
$$H_o = H_L + V_0 \sum_i \delta(z_i - z_0)$$

off-resonant laser interaction

## 4. Anyons & fractional statistics

We adiabatically drag the quasihole around ... and pick up a Berry's phase

- one anyon



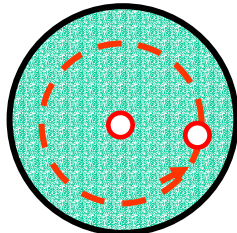
$$\psi_{z_0} \longrightarrow e^{i\phi} \psi_{z_0}$$

Berry's phase

$$\phi = 2\pi n$$

# atoms inside the curve

- two anyons



$$\psi_{z_0 z_1} \longrightarrow e^{i\phi} \psi_{z_0 z_1}$$

✓ there is an extra  $\pi$  phase

✓ independent of area

$$\phi = 2\pi\left(n - \frac{1}{2}\right) = 2\pi n - \pi$$

✓ if we interchange the two holes:  $\pi/2$

$\frac{1}{2}$  anyons:  $\pi/2$   
(fractional statistics)



bosons:  $2\pi$   
fermions  $\pi$

Fractional statistics has never been seen directly!