Systematics of the Relationship between
Vacuum Energy Calculations and Heat Kernel Coefficients

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Let $E(t) = \sum \omega_n e^{-\omega_n t}$ be the vacuum energy associated with the operator H (${\omega_n}^2$ are eigenvalues of H) with an exponential cutoff. As $t \to 0$, $E(t)$ has an expansion

$$E(t) \sim -\frac{1}{2}\frac{\partial}{\partial t}\left[\sum_{s=0}^{\infty} e_s t^{s-d} + \sum_{\substack{s=d+1 \\ s-d \text{ odd}}}^{\infty} f_s t^{s-d} \ln t\right],$$

where d is the spatial dimension; $-\frac{1}{2}e_{1+d}$ is (essentially) the renormalized Casimir energy. If the standard heat-kernel-trace expansion for H is $K(t) \sim \sum b_s t^{(s-d)/2}$, then e_s or f_s is proportional to b_s if $s-d$ is even or negative; when $s-d$ is odd and positive, e_s is undetermined by the bs and depends nonlocally on the geometry of the system. Because the bs are well known, this relationship explains many observations about the presence or absence of various divergent (or logarithmically ambiguous) terms in vacuum energy: (1) The Casimir energy of a spherical shell has such a divergence in even dimensions but not odd (Milton et al.). (2) The Casimir energy for parallel plates has such a divergence (only) for general Robin boundary conditions (Romeo and Saharian). (3) Nonrelativistic Casimir energy vanishes (Milton).