ON THE DIRECT DETERMINATION OF SENSITIVITY, RESOLUTION AND INFORMATION CONTENT OF HELIOSEISMIC DATA – APPLICATION TO THE INVERSION OF THE SOLAR CORE ROTATION RATE.

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ABSTRACT

A resolution analysis was carried out on the inversion matrix for the solar rotation inverse problem. This analysis effectively establishes a direct relationship between the mode set included in the inversion and the spatial resolution, sensitivity and information content of the resulting inferences. We show that such an approach allows us to determine the effect of adding low frequency and low degree p-modes, high frequency and low degree p-modes, and g-modes on the derived rotation rate in the solar radiative zone, in particular the solar core. We concluded that the level of uncertainties that is needed to infer the dynamical conditions in the core when only including p-modes is unlikely to be reached in the near future, and hence sustained efforts are needed towards the detection and characterisation of g-modes.

Key words: Sun: helioseismology - Sun: Oscillations.

1. INTRODUCTION

Over the past decade, increasingly accurate helioseismic observations from ground-based and space-based instruments have given us a reasonably good description of the dynamics of the solar interior [e.g. 10, 11]. Helioseismic inferences have confirmed that the differential rotation observed at the surface persists throughout the convection zone. There appears to be very little – if any – variation of the rotation rate with latitude in the outer radiative zone ($0.4 > r/R_{\odot} > 0.7$). In that region the rotation rate is almost constant (≈ 430 nHz), while at the base of the convection zone, a shear layer – the tachocline – separates the region of differential rotation throughout the convection zone from the one with rigid rotation in the radiative zone.

Despite the large dispersion of the data sensitive to the solar core [see discussion in 4], we can rule out an inward increase or decrease in the solar internal rotation rate down to $r/R_{\odot} \approx 0.25$, by more than 20% of the surface rate at mid-latitude [1, 4, 2]. This is in clear disagreement with the theoretical hydrodynamical models that predict a much faster rotation in the solar core, namely 10 to 50 times faster than the surface rate [e.g. 11].

More recently, [6] and [9] have independently developed new mode fitting procedures to improve the quality of data sensitive to the rotation in the solar core. By using very long time series – in excess of 2000 days – collected with the MDI, GONG and GOLF instruments they have measured rotational splittings for modes with frequencies as low as 1.1 mHz.

In this work, we attempted to establish the sensitivity of helioseismic data sets to the dynamics of inner solar radiative interior, as well as the level of accuracy that helioseismic data should have to resolve the solar core.

2. THEORETICAL BACKGROUND

The starting point of all rotational helioseismic inversion methodologies is the functional form of the perturbation in frequency, $\Delta\nu_{n\ell m}$, induced by the rotation of the sun, $\Omega(r,\theta)$, and given by [see derivation in 8]:

$$\Delta\nu_{n\ell m} = \frac{1}{2\pi}\int_0^R\int_0^\pi K_{n\ell m}(r,\theta)\Omega(r,\theta)drd\theta + \epsilon_{n\ell m} \qquad (1)$$

The perturbation in frequency, $\Delta\nu_{n\ell m}$ with error $\epsilon_{n\ell m}$, that corresponds to the rotational component of the frequency splittings, is given by the integral of the product of a sensitivity function, or kernel, $K_{n\ell m}(r,\theta)$ with the rotation rate, $\Omega(r,\theta)$, over the radius, r, and the co-latitude, θ. The kernels, $K_{n\ell m}(r,\theta)$, are known functions of the solar model.

Equation 1 defines a classical inverse problem for the sun's rotation. The inversion of this set of M integral equations – one for each measured $\Delta\nu_{n\ell m}$ – allows us to infer the rotation rate profile as a function of radius and latitude from a set of observed rotational frequency splittings (hereafter referred as splittings).

The inversion method we used is based on the regularized least-squares methodology (RLS). The RLS method requires the discretization of the integral relation to be inverted. In our case, Eq. 1 is transformed into a matrix relation

$$D = Ax + \epsilon \qquad (2)$$

where D is the data vector, with elements $\Delta\nu_{n\ell m}$ and dimension M, x is the solution vector to be determined at N tabular points, A is the matrix with the kernels, of dimension $M \times N$ and ϵ is the vector containing the errors in D.

The solution is the one that minimizes the quadratic difference $\chi^2 = |Ax - D|^2$, with a constraint given by a smoothing matrix, H, introduced in order to resolve the singular nature of the problem. The general relation to be minimized is

$$S(x) = (Ax - D)^T(Ax - D) + \gamma xHx \qquad (3)$$

where γ is a scalar introduced to give a suitable weight to the constraint matrix H in the solution. Hence, the function x is approximated by

$$x_{\text{est}} = (A^T A + \gamma H)^{-1} A^T D \qquad (4)$$

Replacing D from equation 2 we obtain

$$x_{\text{est}} = (A^T A + \gamma H)^{-1} A^T Ax \stackrel{\text{def}}{=} Rx \qquad (5)$$

hence

$$R = (A^T A + \gamma H)^{-1} A^T A \qquad (6)$$

The matrix R, that combines forward and inverse mapping, is referred to as the resolution or sensitivity matrix [7]. Ideally, R would be the identity matrix, which corresponds to perfect resolution. However, if we try to find a generalized inverse resulting in a resolution matrix R close to the identity, the solution is generally dominated by noise magnification.

In our implementation the inverse matrix is calculated following the Regularized Least Squares technique described in [3]. The individual columns of R displays how anomalies in the corresponding model tabular point are imaged by the combined effect of measurement and inversion. In this sense, each element R_{ij} reveals how much of the anomaly in the j^{th} inversion tabular point is transferred into the i^{th} tabular point. Consequently, the diagonal elements R_{ii} states how much of the information is saved in the model estimate and may be interpreted as the resolvability of x_i.

In the work presenetd here, we used the matrix R to study the sensitivity of helioseismic data sets to the rotation rate of the solar core ($r/R_\odot < 0.25$). We thus present a theoretical analysis on the effect of adding low frequency and low degree p-modes, high frequency and low degree p-modes, and g-modes on the rotation rate of the solar core derived through numerical helioseismic inversion techniques.

3. DESCRIPTION OF HELIOSEISMIC FREQUENCY SPLITTINGS MODE SETS

The physical and dynamical information about the solar core, derived from helioseismic data, is extracted from low and intermediate degree p-modes. [9] analyzed 2088 days-long (starting April 30th 1996 and ending January 17th 2002) time series of SOI/MDI and GONG observations to obtain the largest available mode set for $\ell = 4, 25$, ranging the observed frequencies from 1 to 4.5 mHz. This mode set is complemented with the rotational frequency splittings obtained for $\ell = 1, 3$ using the same 2088 days-long time series of GOLF observations [6].

Fig. 1 shows the observational frequency splittings uncertainties of the combined data set as a function of frequency and mode degree, while Figs. 2 and 3 shows the splittings uncertainties for sectoral modes as a function of the inner turning point of the modes, $(\ell+1/2)/\nu$, and as a function of frequency ν, respectively. These plots clearly illustrate the well known and challenging fact that only a small number of modes penetrate the solar core and that the largest uncertainties are associated with these modes, in particular the $\ell = 1, 2$ modes for frequencies larger than 2 mHz. This is due to the difficulty in separating the effect of rotational splitting from the limited lifetime of the modes.

The long time-series analyzed by [9] allowed the calculation for the first time of rotational splittings for very low frequency modes ($\nu < 1.4$ mHz). The associated small uncertainties of these low frequency modes are bound to improve the inferences about the rotation rate in the radiative interior [5].

4. SENSITIVITY ANALYSIS FOR THE SOLAR CORE

A theoretical analysis was carried out in order to determine the effect of different low degreee mode sets on the derivations of the solar rotation rate of the inner radiative interior. Five different artificial data sets (described in Table 1) were calculated through equation 1 using an artificial rotation rate $\Omega_{art}(r, \theta)$, that is shown in black in Figs. 6 to 10. The observational uncertainties and data noise were taken from the actual mode sets calculated by [9] and [6].

Table 1. Description of the artificial data sets used to study the sensitivity of helioseismic data to the dynamics of the solar core.

		Freq. range (mHz)		
Data set	note	$\ell = 1$	$\ell = 2, 3$	$\ell > 3$
Data set 1		1 - 2	1 - 2	> 1
Data set 2		1 - 2	> 1	> 1
Data set 3		1 - 3	> 1	> 1
Data set 4	1	1 - 3	> 1	> 1
Data set 5	2	0.1 - 2	0.1 - 2	> 1

[1] Data set 4 differs from data set 3 in the observational error distribution, being the errors for data set 4 taken from the sectoral splittings of $\ell = 25$ modes.
[2] Two $\ell = 1$ and two sectoral $\ell = 2$ g-modes rotational splittings were added to data set 5.

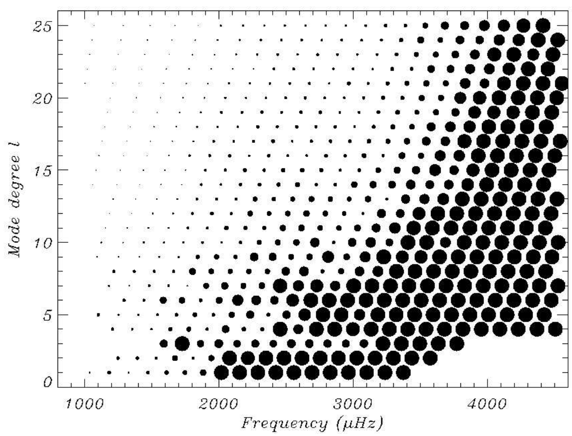

Figure 1. Mode degree as a function of frequency ν for a combined GOLF + MDI frequency splittings set [6, 9]. The splittings were obtained from 2088 day-long time-series. The size of the circles is proportional to the observational errors of the frequency splittings. For illustrating purposes errors larger than 15 *nHz were assigned the same circle size.*

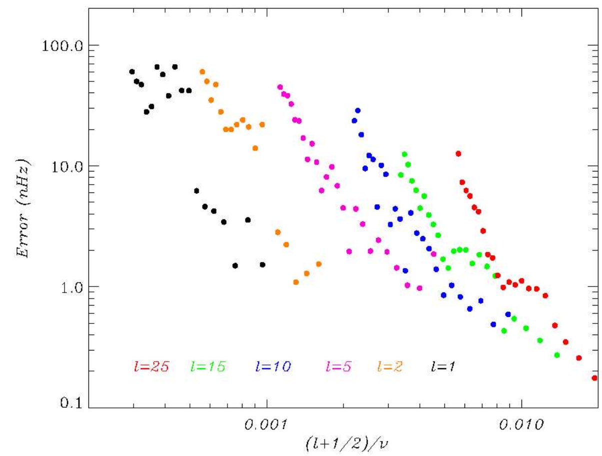

Figure 2. Observational errors for a sample of sectoral frequency splittings as a function of the inner turning point $(\ell + 1/2)/\nu$.

The resolution matrix R was calculated for the five artificial data sets, as illustrated in Fig. 4, where the diagonal elements of R for the rotation rate at the equator are presented. Data sets 1, 2 and 3 are nearly insensitive to the rotation of the solar core. Data set 4 contains the same mode set than data set 3, but it is sensitive to the solar core because the observational errors are much lower in data set 4. The addition of a few g-modes (data set 5) increases significantly the sensitivity to the solar core.

It is important to notice that even with the addition of g-modes, there is not latitudinal sensitivity at the solar core, as illustrated in Fig 5, where the diagonal elements of R for the rotation rate at five different co-latitude ranges are presented. Sensitivity to the equatorial regions of the solar core is significantly larger than the sensitivity to other co-latitudes.

The conclusions derived from Figs. 4 and 5 could be also derived from the numerical inversions of the data sets 1 to 5, as illustrated in Figs. 6 to 10. Only in the cases of data sets 4 and 5, is it possible to infer the main trends of the rotation rate below $r/R_{\odot} \approx 0.15$; however, it was necessary to include, on one hand g-modes (data set 5) modes that have yet to be observed, and on the other hand data with unrealistic small observational errors (data set 4). In the latter case, the most likey way to reduce observational errors consists of increasing the length of the time series. Fig. 11 shows the distribution of observational errors associated to $\ell = 25$ sectoral modes for five 728 days-long data sets and the 2088 days-long data set, all calculated by [9]. The formal observational errors are proportional to the square root of the length of the time series, and hence it could be necessary to observe for decades to reduce the observational errors calculated for very low degree and low frequency modes to the present levels cal-

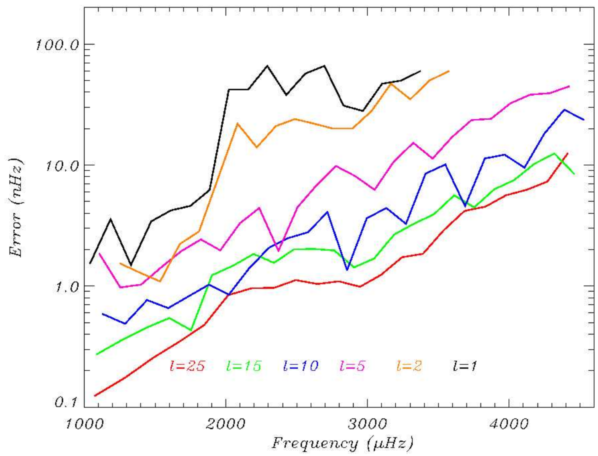

Figure 3. Observational errors for a sample of sectoral frequency splittings as a function of the mode frequency, ν.

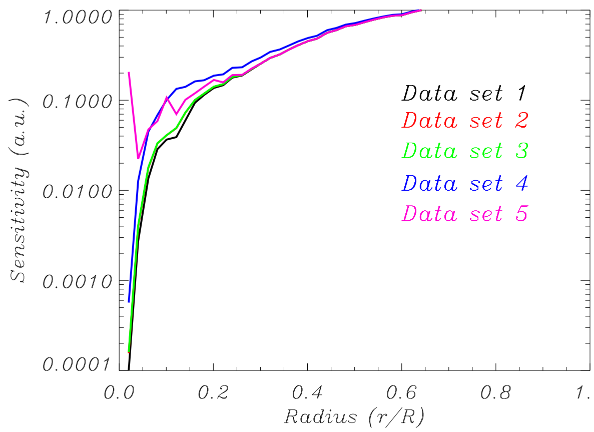

Figure 4. Sensitivity of the different data sets that are presented in Table 1 to the rotation rate of the solar interior.

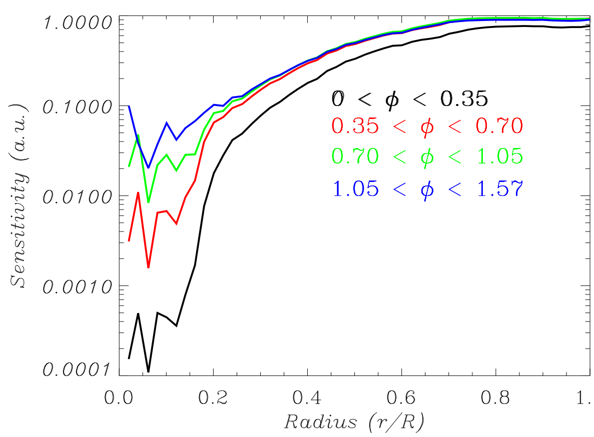

Figure 5. Sensitivity of data set 5 to the solar core at different latitudes. Values for co-latitude ranges are given in radians, the equator being at $\pi/2$ radians.

culated for $\ell = 25$ modes.

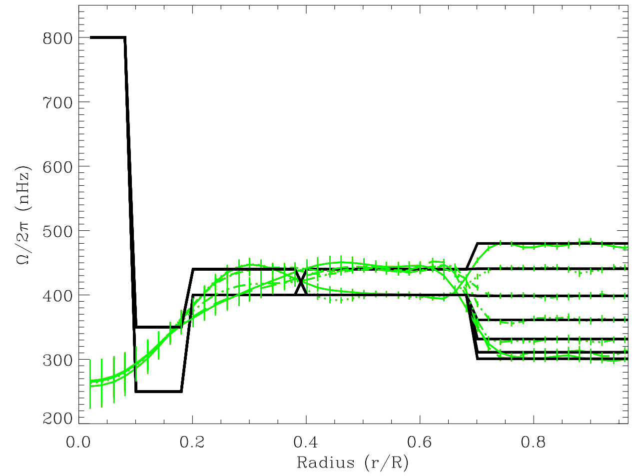

Figure 6. Numerical inversion of data set 1 (in green) as a function of radius for a set of co-latitudes. The artificial rotation rate used to calculate data set 1 is shown in black.

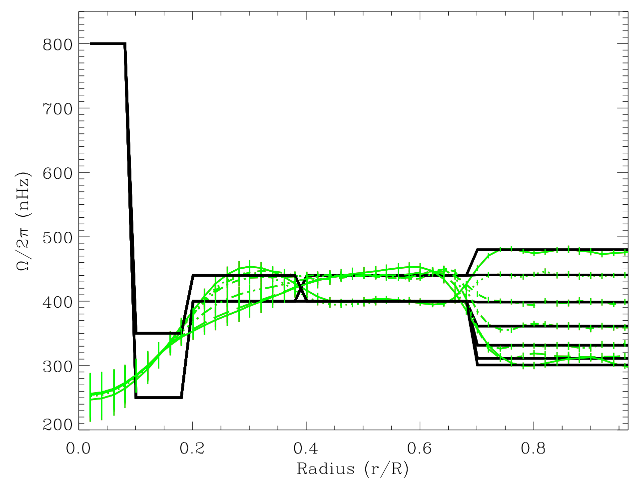

Figure 7. As in Fig. 6, but for data set 2.

5. CONCLUSIONS

The calculation of the resolution matrix R is a rapid and intuitive way of evaluating the sensitivity of helioseismic data to the dynamics of the solar interior, in particular in the core ($r/R_\odot < 0.25$). We conclude that with the present accuracy of the available splittings, it is not possible to derive the dynamical conditions below $r/R_\odot \approx 0.2$. This results from the relatively large observational uncertainties of the modes sensitive to the solar

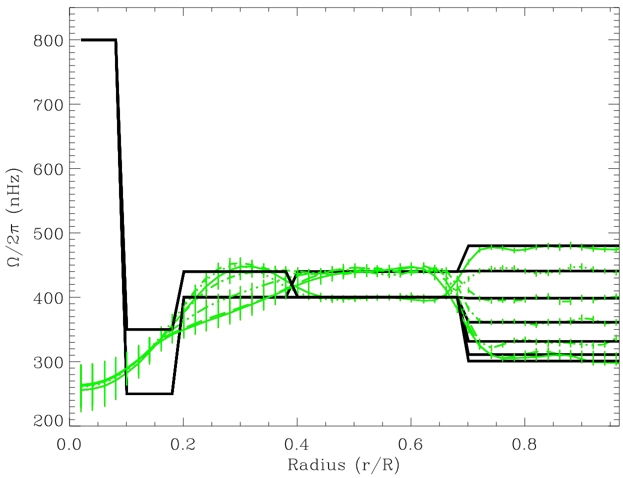

Figure 8. As in Fig. 6, but for data set 3.

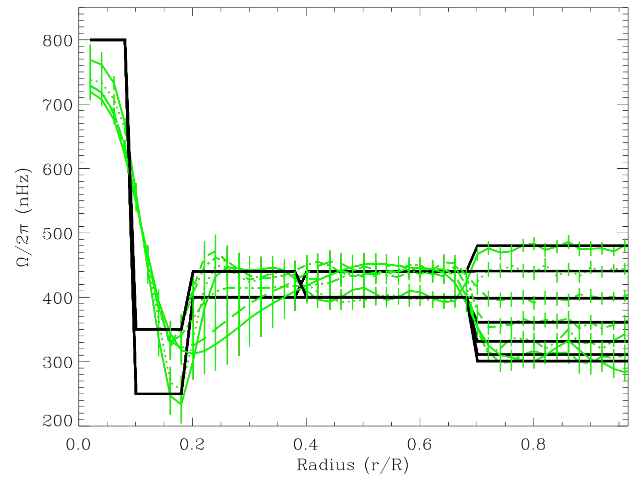

Figure 10. As in Fig. 6, but for data set 5.

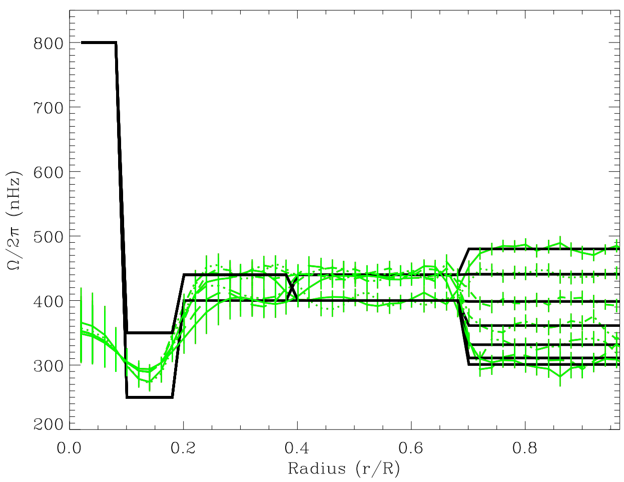

Figure 9. As in Fig. 6, but for data set 4.

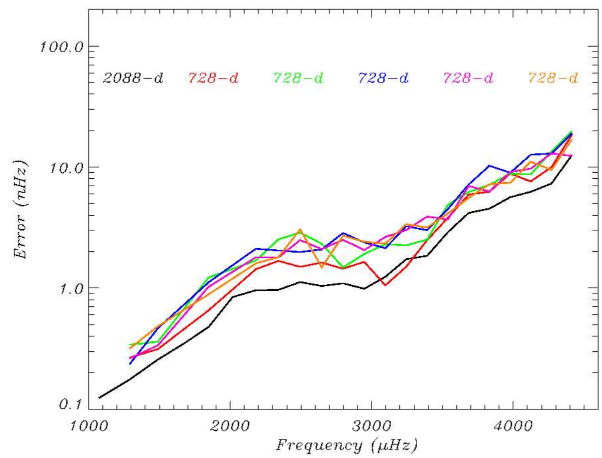

Figure 11. Observational errors for $\ell = 25$ sectoral frequency splittings as a function of the mode frequency, ν. Different colours reflect the length of the time-series the data were calculated from.

core, in particular low degree and high frequency modes. The level of uncertainties that is needed to infer the dynamical conditions in the core when only including p-modes is unlikely to be reached in the near future, and hence sustained efforts are needed towards the detection and characterisation of g-modes.

ACKNOWLEDGMENTS

The Solar Oscillations Investigation - Michelson Doppler Imager project on SOHO is supported by NASA grant NAS5–3077 at Stanford University. SOHO is a project of international cooperation between ESA and NASA.

The GONG project is funded by the National Science Foundation through the National Solar Observatory, a division of the National Optical Astronomy Observatories, which is operated under a cooperative agreement between the Association of Universities for Research in Astronomy and the NSF.

This work was funded by the grant AYA2004-04462.

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