Annenberg/CPB
Professional Development Workshop Guide

PRIVATE UNIVERSE PROJECT IN MATHEMATICS

A professional development workshop
for K-12 mathematics teachers

Produced by the Harvard-Smithsonian Center for Astrophysics in collaboration with the Robert B. Davis Institute for Learning at Rutgers University

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ISBN: 1-57680-348-1

Funding for
Private Universe Project in Mathematics
is provided by Annenberg/CPB.

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About the Workshop

Overview

This workshop provides an interactive forum for teachers, administrators, and other interested adults to explore issues about teaching and learning mathematics. Central to each session is a 60-minute videotape that offers a sequence of episodes that show children and teachers engaged in authentic mathematical activity and discussion, consistent with state and national standards for teaching and evaluating mathematics. These episodes come from a variety of sources in diverse school communities and across grade levels from pre-kindergarten through grade 12. The episodes and accompanying narratives in each videotape focus on:

1. students and teachers actively engaged in doing mathematics;
2. conditions that encourage meaningful mathematical activity; and
3. implications for learning, teaching, and assessment.

The materials and activities presented in the sessions have been developed in long-term research programs about mathematical thinking that share certain presuppositions about learning and teaching. Key to this perspective is that knowledge and competence develop most effectively in situations where students, frequently working with others, work on challenging problems, discuss various strategies, argue about conflicting ideas, and regularly present justifications for their solutions to each other and to the entire class. The role of the teacher includes selecting and posing the problems, then questioning, listening, and facilitating classroom discourse, usually without direct procedural instruction.

Each videotape contains episodes from a 12-year research study carried out in the Kenilworth, New Jersey public schools through a partnership with the Robert B. Davis Institute for Learning at Rutgers University. This partnership included a classroom-based staff development program in mathematics for teachers and administrators at the K-8 Harding Elementary School. Classroom sessions in which students frequently worked together in small groups on meaningful problem activities were videotaped as a regular part of the project. One classroom of children was followed from first through third grades by regularly videotaping small-group problem-solving sessions, whole class discussions, and individual task-based interviews. From 1991 to the present, with periodic support from two National Science Foundation grants, the research team has continued documenting and studying the thinking of a focus group of these children through grade 12. The videotapes from these later years document the students as they participated in problem-based activities developed by the university researchers in classroom lessons, after-school sessions, a two-week summer institute, and individual and small-group task-based interviews.

In the workshop videotapes, participants will see some of the same children solving mathematical problems at various grade levels over the years. In addition, the activities they engage in have been used in other communities and at other grade levels. In June 2000, the focus group of students in this study graduated from Kenilworth's David Brearley Middle/High School.

Workshop participants will explore particular questions about learning and teaching mathematics based on the shared experience of watching the videotapes. Key questions are:

1. How do children (and adults) learn mathematics?
2. How do children (and adults) learn to communicate about mathematics and to explain and justify solutions to problems?
3. What conditions—environments, activities, and interactions—are most helpful in facilitating this development?

About the Workshop

Overview, cont'd.

4. What does it mean to be a teacher of mathematics?
5. What is the connection between learning and teaching?

Each videotape includes episodes of children engaged in mathematical problem-solving. The goal is for participants to become able to recognize what is mathematical in students' activity by attending very closely to what they do and say. As they observe, study, and discuss what they see on the tape from the perspective of the questions listed above, participants will gain insights about learning and teaching. In preparation, participants need to build their own solutions to the central problems in each tape with assignments given between sessions and during the first hour of each workshop. They are encouraged to select and use appropriate problems with their own students and read further about the learning and teaching of these ideas.

Workshop Descriptions

Workshop 1. Following Children's Ideas in Mathematics

An unprecedented long-term study conducted by researchers at Rutgers University followed the development of mathematical thinking in a randomly selected group of students for 12 years—from first grade through high school—with surprising results. In an overview of the study, we look at some of the conditions that made the students' math achievement possible.

Workshop 2. Are You Convinced?

Proof making is one of the key ideas in mathematics. Looking at teachers and students grappling with the same probability problem, we see how two kinds of proof—proof by cases and proof by induction—naturally grow out of the need to justify and convince others.

Workshop 3. Inventing Notations

We learn how to foster and appreciate students' notations for their richness and creativity. We also look at some of the possibilities that early work in creating notation systems might open up for students as they move on toward algebra.

Workshop 4. Thinking Like a Mathematician

What does a mathematician do? What does it mean to "think like a mathematician"? This program parallels what a mathematician does in real life with the creative thinking of students.

Workshop 5. Building on Useful Ideas

One of the strands of the Rutgers long-term study was to find out how useful ideas spread through a community of learners and evolve over time. Here, the focus is on the teacher's role in fostering thoughtful mathematics.

Workshop 6. Possibilities of Real-Life Problems

Students come up with a surprising array of strategies and representations to build their understanding of a real-life calculus problem—before they have ever taken calculus.

Workshop 7. Next Steps (required for those seeking graduate credit)

Participants will review key ideas presented during the workshops and consider implications for their own teaching.

About the Workshop

Video Clip Descriptions

Workshop 1. Following Children's Ideas in Mathematics

Part 1—The Youngest Mathematicians

5 min. Mathematics in Free Play?
Prof. Herbert Ginsburg, a psychologist at Columbia University Teachers College, finds that when you examine what children at ages three and four actually do in free play, more than 50 percent of the time they are engaged in mathematical tasks.

5 min. The Beginning of the Rutgers/Kenilworth Long-Term Study
In 1987, Prof. Carolyn Maher, from the Robert B. Davis Institute for Learning at Rutgers University, is invited to the elementary school in Kenilworth, New Jersey, a small town of mostly moderate-income working families. What begins as a professional development project for teachers evolves into a 12-year study—carefully documented on video—of the development of mathematical thinking of a randomly selected group of students.

12 min. Shirts and Pants
Video clips from the research archive show how the students approach a combinations problem. During their first attempt, in the second grade, they come up with a variety of answers. Four months later, in the third grade, they spontaneously arrive at the correct answer.

5 min. Cups, Bowls, and Plates
Later in the third grade, in an extension of the Shirts and Pants problem, one student, Stephanie, shows how she used multiplication to arrive at an answer.

Part 2—From "Towers" to High School

10 min Building Towers Five-High
The Kenilworth students in the fourth grade are seen working on the Towers problem ("How many different towers can you build by selecting from blocks of two colors?") juxtaposed with clips of the same students, now seniors in high school, reflecting back on the experience.

5 min. Ninth-Grade Geometry
Transferred to a regional school, where the math teacher's focus was to finish every chapter in the textbook, the students feel they learned very little.

5 min. Romina With Her Parents
Now in her senior year, one of the students in the focus group, Romina, confers with her parents about colleges.

5 min. Romina at School
In her AP Calculus class, Romina solves a problem in an interesting way that surprises her teacher.

About the Workshop

Video Clip Descriptions, cont'd.

Workshop 2. Are You Convinced?

Part 1—Teachers Building Proofs

25 min. Englewood, New Jersey—Teachers Workshop
Englewood, a town with unsatisfactory student test scores, is implementing a long-term project to improve math achievement. As part of a professional development workshop designed in part to give K-8 teachers more confidence in their own mathematical thinking, teachers come up with a wide variety of justifications for their answers to the Towers problem.

Part 2—Students Building Proofs

8 min. Working With Towers
In the third grade, students in the Kenilworth study build towers four-high, and hypothesize about towers three-high. In the fourth grade, they build towers five-high.

20 min. "Gang of Four"
In the fourth grade, a group of four students from the Kenilworth focus group come up with mathematically sound proofs for the number of combinations in the Towers problem—and then generalize their solution to apply to all towers.

Workshop 3. Inventing Notations

Part 1—Putting It on Paper: Elementary Students Invent Notations

15 min. Pizzas in the Classroom
In Englewood, New Jersey, Blanche Young, who attended the summer workshop, tries out one of the problems with her fourth-grade students. Later, she meets with Arthur Powell to discuss the lesson.

5 min. New Brunswick, New Jersey
In addition to extensive research in Kenilworth, New Jersey, the math researchers from the Robert B. Davis Institute for Learning also visit the urban district of New Brunswick. Here, a group of fifth-graders comes up with a number of useful notations to help solve the Pizza problem.

5 min. Brandon Connects Pizzas and Towers
In Colts Neck, New Jersey, one student, Brandon, explains the correspondence between the Pizza and Towers problems and describes in detail how the same notation can be used for both.

Part 2—Notations Evolve As Students' Thinking Evolves (and Vice Versa)

25 min. Kenilworth Study: Pizzas
In the fourth grade, the students encounter counting problems where the solutions cannot be built using standard manipulatives. As he invents his own notation systems, one student, Matt, builds on previous work to arrive at a solution for an even more complex problem: Pizzas With Halves.

About the Workshop

Video Clip Descriptions, cont'd.

Workshop 4. Thinking Like a Mathematician

Part 1—Strategies for Solving Problems

10 min. How a Mathematician Approaches Problems
Fern Hunt, a mathematician at the National Institute for Science and Technology, is seen as she collaborates with colleagues to solve difficult technical problems. Using the metaphor of the children's game Towers of Hanoi, she explains her approach to solving problems.

15 min. Towers of Hanoi
In a research session conducted by the late Robert Davis of Rutgers University, sixth-graders from the Kenilworth study put into practice problem-solving strategies that mirror the strategies outlined by Fern Hunt.

Part 2—Encouraging Students To Think

10 min. Provo, Utah—High School Algebra
Presidential award-winning math instructor Janet Walter has inherited a ninth-grade Algebra I class part-way through the school year. How can she overcome their reticence to share ideas, and help them think for themselves?

5 min. Revisiting Problems After Five Years
Kenilworth 10th graders re-examine the same problem they had last seen in the fifth grade—the Pizza problem. One student, Michael, uses the binary number system to his advantage.

10 min. Romina's Proof
Responding to a problem posed by one of the students, Romina, a 10th-grader, invents a proof solution and shares it with the others.

Workshop 5. Building on Useful Ideas

Part 1—The Changing Role of the Teacher: Elementary Classrooms

5 min. Englewood—Kindergarten: Stacking Blocks
In Englewood, New Jersey, a kindergarten teacher from a summer workshop uses blocks as mathematical objects in an addition activity.

5 min. Englewood—Second Grade: Probing Student Thinking
How can a teacher know what an individual student is thinking when there are 24 or more students in the room? In Englewood, a second-grade teacher tries to follow her students' thinking by asking appropriate questions as she moves from table to table.

10 min. Englewood—Fourth Grade: Towers
Fourth-grade teacher Blanche Young attempts the Towers activity for the first time with her students. She feels that their work is valuable, but questions how much time these open-ended activities are taking away from the standard curriculum.

About the Workshop

Video Clip Descriptions, cont'd.

Workshop 5, cont'd.

5 min. "Equations"
In Colts Neck, New Jersey, fourth-grade teacher and former Rutgers researcher Amy Martino finds out that what started as a 15-minute "warm-up" question evolves into an interesting discussion about equations.

Part 2—Pascal's Triangle and High School Algebra

10 min. Jersey City: Ice Cream Problem
Algebra II teacher Gina Kiczek introduces a problem that helps her students learn the difference between permutations and combinations.

5 min. What Is Pascal's Triangle?
An overview of the "Arithmetic Triangle": what it is, its history, and how it is linked to the Towers and Pizza problems.

10 min. World Series Problem
In the 11th grade, the Kenilworth students build on their thinking as young children to tackle a complex—and realistic—probability problem.

5 min. *n* choose *r*: Moving Toward Standard Notation
In an after-school research session, Mike derives the formula for the general combinatorics problem "compute *n* choose *r*."

Workshop 6. Possibilities of Real-Life Problems

Part 1—The Catwalk: Representing What You Know

25 min. The Catwalk, Part 1: Representing What You Know
In a voluntary two-week summer workshop, 18 high school seniors from Kenilworth and New Brunswick work on a real-life problem based on a sequence of 24 photographs of a cat in motion. The question, "How fast is the cat moving in frame 10 and frame 20?," deals with some of the fundamental ideas of calculus. Students find several ways to represent their growing understanding: visual, symbolic, and kinesthetic.

Part 2—Betting on What You Know

25 min. The Catwalk, Part 2: Betting on What You Know
Continuing the problem to its conclusion, the students use their representations as the basis for reconstructing the cat's movement.

5 min. The Class of 2000 and Beyond
In Kenilworth, New Jersey, the students in the Kenilworth study are graduating from high school. In a montage of interviews, they reflect back on their involvement with the long-term study, look toward the future, and wonder, "Does the way in which you study math make a difference?"

Workshop Components

On-Site Activities and Timelines

Getting Ready (Site Investigation)

In preparation for watching the video, you will engage in 60 minutes of doing math problems, discussion, and activity—exploring the math for yourself.

Watch the Workshop Video

Watch the 60-minute workshop video, which is divided into two parts. Each part includes a focus question that appears on screen and can be found in this guide. Participants in workshop settings will either view the live broadcast on the Annenberg/CPB Channel or watch a pre-taped video of the program. If you are watching a live broadcast, discuss the focus questions at the end of the broadcast. If you are watching on videotape, stop the tape for a discussion at the end of each part.

Going Further (Site Investigation)

Wrap up the workshop with an additional 30 minutes of investigation through discussion and activity. The Episode Boxes in this guide contain brief descriptions of video clips and related questions for exploring learning and teaching.

For Next Time

Homework Assignments

You will be assigned exercises or activities that tie into the previous workshop or prepare you for the next one.

Reading Assignments

Readings will be assigned to prepare you for the following workshop. They can be found in the Appendix of this guide. Some are also available online at www.learner.org/channel/workshops.pupmath.

Ongoing Activities

You may want to carry on these activities throughout the course of the workshop.

Keep a Journal

Participants are encouraged to keep a reflective journal throughout the workshop, to keep track of reactions to readings and videotapes, to collect and reflect on data, and to record teaching ideas for yourself.

Visit the Web Site: www.learner.org/channel/workshops/pupmath

Go online for additional activities, resources, and discussion opportunities.

Share Ideas on Channel-TalkPUPMath

Participants may subscribe to an email discussion list and communicate with other workshop participants online. To subscribe to Channel-TalkPUPMath, visit: http://www.learner.org/mailman/listinfo/channel-talkpupmath.

About the Contributors

Carolyn Maher

Carolyn Maher is a professor of mathematics education in the Graduate School of Education at Rutgers University and the director of the Robert B. Davis Institute for Learning. The Davis Institute has a successful history of long-term commitments to education reform initiatives and works closely with schools and districts in New York and New Jersey. Dr. Maher's longitudinal research, now in its 12th year, focuses on the development of children's mathematical thinking and development of proof. She has given presentations and led workshops for groups of teachers, math educators, and administrators throughout the United States as well as in diverse settings such as Australia, Brazil, Canada, Israel, South Africa, Mozambique, Japan, and China. Dr. Maher is also the editor of *The Journal of Mathematical Behavior* and the director of the regional center at Rutgers University for the New Jersey Statewide Systemic Initiative.

Alex Griswold

Alex Griswold is a producer and videographer at the Harvard-Smithsonian Center for Astrophysics. Mr. Griswold has been producing educational films and television programs for over 25 years, and has taught video and film production at Harvard University; the Department of Defense Dependents Schools, Madrid, Spain; and the Boston Film/Video Foundation. Since joining the staff of the Center for Astrophysics in 1992, he has specialized in the creation of teacher education materials in mathematics and science, gaining wide experience working with children and teachers in educational settings. His current responsibilities include management of a science visualization laboratory and producing interactive media, video, and film that enhance learning in math and science.

Alice Alston

Alice Alston is a senior mathematics education specialist at the Robert B. Davis Institute for Learning and a visiting associate professor of mathematics education at Rutgers University Graduate School of Education. She co-authored the PACKETS Program for Upper Elementary Mathematics while working at the Educational Testing Service in Princeton, New Jersey. Dr. Alston works extensively with teachers in New Jersey schools, and particularly urban schools. She has additional expertise in standards-based professional development for mathematics educators as a result of experience implementing the NSF Middle Grades Project, Linking Instruction and Assessment. She leads a professional development program in mathematics, science, and literacy in three urban districts in New Jersey. Formerly, Dr. Alston taught middle and high school and was chair of the Middle School Mathematics Department at Princeton Day School.

Emily Dann

Emily Dann is a senior mathematics education specialist at the Robert B. Davis Institute for Learning. She has taught mathematics at the middle school through college level, and mathematics education at both undergraduate and graduate levels. Dr. Dann has worked in both the New York and New Jersey Statewide Systemic Initiatives and provides expertise in implementing standards-aligned, research-based curriculum programs for K-12 mathematics educators. She currently coordinates the Rutgers-Colts Neck Partnership for Implementing a Thinking Curriculum, a professional development project funded by the Exxon Mobil Foundation. Dr. Dann is the associate director of the regional center at Rutgers University for the New Jersey Statewide Systemic Initiative.

About the Contributors

Regina Kiczek

Regina Kiczek is K-8 mathematics supervisor for the Westfield, New Jersey school district. A former high school mathematics teacher with over 25 years of teaching experience, Ms. Kiczek has recently completed her doctorate in mathematics education at Rutgers University. She has been a research team member of the NSF-funded longitudinal study of students' proof making, and has experience planning and implementing professional development programs for K-12 mathematics educators. Her research into the development of probabilistic thinking of students was presented in 2000 at the Ninth International Congress on Mathematics Education, Tokyo, Japan.

Arthur Powell

Arthur Powell teaches mathematics and mathematics education in the Department of Education and Academic Foundations at Rutgers University. For over two decades, Professor Powell has worked with elementary and secondary teachers in the United States, Mozambique, Brazil, and Canada. He is a faculty research scientist at the Robert B. Davis Institute for Learning and currently directs a teacher development project for the district of Englewood, New Jersey, working with teachers in summer workshops and supporting teachers' work with children in their classrooms.

Robert Speiser

Robert Speiser, whose work as a mathematician has been recognized internationally, is currently a professor of mathematics education at Brigham Young University. He has taught elementary school and worked with teachers at all levels. Dr. Speiser leads a K-3 Mathematics Specialist Project which supports the joint efforts of university mathematics educators, a study group of elementary school teachers, and students in a university-led teacher preparation program. His research in education concentrates on the growth of mathematical understanding, especially through explorations of rich tasks in settings which promote reflection and discussion. Dr. Speiser is the editor of *The Journal of Mathematical Behavior.*

Elena Steencken

Elena Steencken is a mathematics education specialist and assistant director of the Robert B. Davis Institute for Learning in the Graduate School of Education at Rutgers University. She is currently completing her doctorate in mathematics education and has worked extensively in preservice elementary and secondary teacher preparation programs at Rutgers University. She has also collaborated with K-12 inservice teachers in various professional development programs in New Jersey and New York. She has recently prepared *Exploring To Build Meaning About Fractions,* a unit booklet produced by the Robert B. Davis Institute for Learning for use by preservice and inservice teachers.

Charles Walter

Charles Walter, a mathematician, is currently a professor of mathematics education at Brigham Young University. His 30-year commitment to mathematics and mathematics education encompasses both the preparation of elementary and secondary teachers and the collaboration with teachers and children in classrooms. He leads a K-3 Mathematics Specialist Project which supports the joint efforts of university mathematics educators, a study group of elementary school teachers, and students in a university-led teacher preparation program. Dr. Walter has designed and conducted NSF-sponsored workshops centered on mathematics and pedagogy in secondary and post-secondary calculus classrooms. His research focuses on how learners, especially children, build and represent mathematical knowledge. He is the assistant editor of *The Journal of Mathematical Behavior.*

Helpful Hints

Successful Site Investigations

Included in the materials for each workshop, you will find detailed instructions for the content of your **Getting Ready** and **Going Further** Site Investigations. The following hints are intended to help you and your colleagues get the most out of these pre- and post-video discussions.

Designate a Facilitator

Each week, one person should be responsible for facilitating the Site Investigations (or you may select two people—one to facilitate Getting Ready, the other to facilitate Going Further). The facilitator does not need to be the Site Leader, nor does the role need to be held by the same person each week. We recommend that participants rotate the role of facilitator on a weekly basis.

Review the Site Investigations and Bring the Necessary Materials

Be sure to read over the Getting Ready and Going Further sections of your materials before arriving at each workshop. The Site Investigations will be the most productive if you and your colleagues come to the workshops prepared for the discussions. The weekly readings and homework assignments also provide for productive and useful workshop discussions. A few of the Site Investigations require special materials. The facilitator should be responsible for bringing these when necessary.

Keep an Eye on the Time

You should keep an eye on the clock so that you are able to get through everything before the workshop video begins. In fact, you may want to set a small alarm clock or kitchen timer before you begin the Getting Ready Site Investigation to ensure that you won't miss the beginning of the video. (Sites that are watching the workshops on videotape will have more flexibility if their Site Investigations run longer than expected.)

Record Your Discussions

We recommend that someone take notes during each Site Investigation, or even better, that you make an audiotape recording the discussions each week. These notes and/or audiotape can serve as "make-up" materials in case anyone misses a workshop.

Share Your Discussions on the Internet

The Site Investigations are merely a starting point. We encourage you to continue your discussions with participants from other sites on the Teacher Talk area of the Web site at www.learner.org/channel/workshops/pupmath and on Channel-TalkPUPMath, the workshop email discussion list.

Materials Needed

Site Investigations

Workshop 1. Following Children's Ideas in Mathematics

Paper, pens, or markers: The group will need paper and pens or markers for preparing solutions to the problems. Especially if you are a large group, you may want to have an overhead projector, blank transparencies, and pens for participants to use when sharing solutions.

Unifix® or other snap cubes: Each participant will need about 100 cubes (50 each of two colors) to complete the homework assignment for Workshop Two. Although not essential, sharing and discussing solutions will be much easier if everyone is using the same two colors. If this is impossible, sets of cubes need to be made up for each participant to use with two colors that can be designated as "light cubes" and "dark cubes" when their solutions are discussed. If Unifix® cubes are not available, use the "cut-out cubes" sheets included in this guide at the end of Workshop 1.

Workshop 2. Are You Convinced?

Unifix® or other snap cubes, or "cut-out cubes" (see above)

Workshop 5. Building on Useful Ideas

Pascal's Triangle Worksheet (see pages 48-49 of this guide).

Workshop 6. Possibilities of Real-Life Problems

Catwalk: Each participant will need at least two copies of the cat photographs on 11 x 17 paper and on transparencies (a copy is included on pages 56-57 of this guide; please piece together the two parts and photocopy onto 11 x 17 paper); metric rulers (clear plastic ones work best); graph paper; a calculator (graphing calculator if possible); and pens or markers for preparing solutions to the problems. You will also want to have an overhead projector, blank transparencies, and pens for sharing solutions.

Workshop 1.

Following Children's Ideas in Mathematics

How far can children's natural curiosity and interest in math take them? How do students do mathematics before they are taught procedures for solving problems? Dr. Carolyn Maher and a team of researchers from Rutgers University are pioneers in the effort to answer these questions. Their work documents how students approached a progressive series of challenging mathematics problems beginning in the first grade and continuing all the way through high school. This unprecedented study provides compelling evidence that, given the right conditions, students can accomplish amazing things in mathematics. In Workshop 1, we look at an overview of Dr. Maher's work and its impact on a group of students from Kenilworth, New Jersey.

Part 1—The Youngest Mathematicians

Looking through the lens of developmental psychologist Dr. Herbert Ginsburg, what looks like play in preschool children is actually the beginning of serious mathematical thinking. Elementary school classrooms in Kenilworth provide a starting point for the Rutgers researchers to track how mathematical thinking develops as it begins to be formally introduced. We hear how Dr. Maher's professional development work with teachers grew into a long-term study and we observe small groups of children begin to tackle a combinations problem called Shirts and Pants. Students then progress to a more complicated problem: Cups, Bowls, and Plates. Archival footage shows how the students worked toward a solution over time by devising their own way of representing and solving the problem.

On-Screen Participants: Dr. Herbert Ginsburg, Columbia University; Dr. Carolyn Maher, Rutgers University; Dr. Amy Martino, Rutgers University. **Student Participants:** Dana, Michael, and Stephanie, Kenilworth, New Jersey Public Schools, Grades 2 and 3.

Part 2—From "Towers" to High School

Continuing a chronological history of the long-term study, Dr. Maher introduces the Towers problem. The activity, which investigates how many different five-cube-high towers can be made by selecting from blocks of two colors, would be at home in any high school or college probability class. The difference is that Dr. Maher gives the students objects, not formulas, and asks them to justify their results. Students jump on the problem with enthusiasm, as the researchers purposefully step back to "see what happens." As the study continues, the students take on more complex problems. However, during this time, the students in the focus group are also being affected by the school and community environment. In the ninth grade, the local school is closed, and the students are assigned to a geometry class where rote learning and memorization are the order of the day.

On-Screen Participants: Dr. Carolyn Maher, Rutgers University; Dr. Amy Martino, Rutgers University. **Student Participants:** Brian, Dana, Jeff, Michael, Michelle, Romina, and Stephanie, Kenilworth, New Jersey Public Schools, Grade 4.

On-Site Activities and Timeline

60 minutes

Getting Ready

Try the following activities for yourself in preparation for watching the workshop video.

1. Shirts and Pants

a. Solve the problem below for yourself and develop a way to convince others that your solution is correct.

Stephen has a white shirt, a blue shirt, and a yellow shirt. He has a pair of blue jeans and a pair of white jeans. How many different outfits can he make?

b. Share and compare your solution and justification with others in your group.

c. Imagine giving this problem to students that you teach. Do you think it is an appropriate problem for your students? How do you think your students would solve and justify the problem?

d. Solve the following extensions to the Shirts and Pants problem. Share your solutions.

1. Adding an Item

Remember that Stephen has a white shirt, a blue shirt, and a yellow shirt. He has a pair of blue jeans and a pair of white jeans. Stephen also has a brown belt and a black belt. How many different outfits can he make now?

2. Reversibility

Mario has exactly 16 different outfits. Decide how this might be possible. Specify what pieces of clothing he might have.

2. Cups, Bowls, and Plates

a. Solve the following problem.

Pretend that there is a birthday party in your class today. It's your job to set the places with cups, bowls, and plates. The cups and bowls are blue or yellow. The plates are either blue, yellow, or orange. Is it possible for 10 children at the party to each have a different combination of cup, bowl, and plate?

Is it possible for 15 children at the party each to have a different combination of cup, bowl, and plate?

b. Compare and justify your solutions with the group.

c. Reflect on whether this is a problem you might use with your students. How do you think that your students would solve this problem?

On-Site Activities and Timeline

60 minutes

Watch the Workshop Video

Part 1

On-Screen Math Activities

Shirts and Pants
Stephen has a white shirt, a blue shirt, and a yellow shirt. He has a pair of blue jeans and a pair of white jeans. How many different outfits can Stephen make?

Cups, Bowls, and Plates
There is to be a birthday party for one of the students in class. There are blue and yellow cups, blue and yellow bowls, and blue, yellow, and orange plates. Can 10 children have a different combination of cups, bowls, and plates?

Focus Question

We've just seen the teacher/researchers repeating the same, or similar, problems three times over the course of almost a year. What can we say about the changes in the students' methods over time?

Part 2

On-Screen Math Activities

Towers
Build all possible towers that are five (or four, or three, or *n*) cubes high by selecting from plastic cubes in two colors. Provide a convincing argument that all possible arrangements have been found.

Focus Question

We've had a chance to look at an overview of the Rutgers' long-term study. In what ways are the students in the study similar to your students? How are they different?

On-Site Activities and Timeline

30 minutes

Going Further

Please address as many of these questions as possible during your allotted time and consider the remainder as part of your homework.

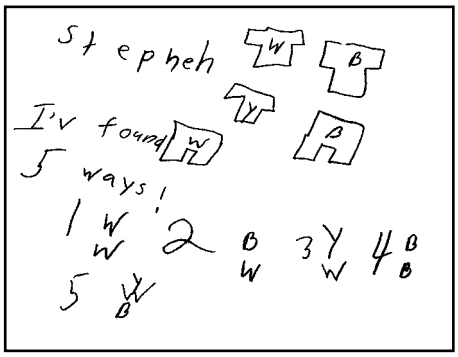
Episode Box One: Mathematical Thinking Among the Preschool Children

Professor Herbert Ginsburg described the activity of preschool children to be mathematical when it involved ideas like numbers, size comparison, measurement, and patterns.

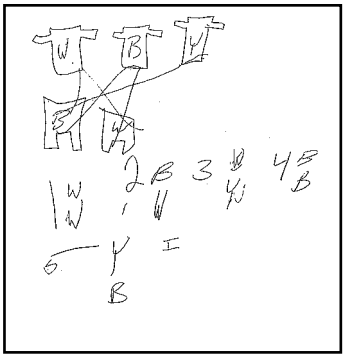
- What mathematics did you see the preschool children doing?

Episode Box Two: Shirts and Pants in Grade 2

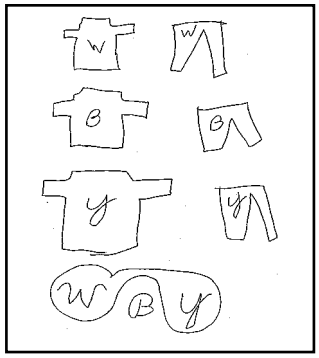
Stephanie, Dana, and Michael each solved the Shirts and Pants problem. They had different solutions. Although Stephanie and Dana both had 5 outfits, they were different. Michael's solution had 3 outfits, including his own pair of yellow pants.



Stephanie's Solution



Dana's Solution



Michael's Solution

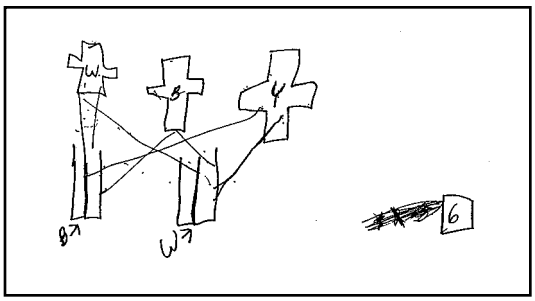
- Discuss the children's problem solving.

On-Site Activities and Timeline

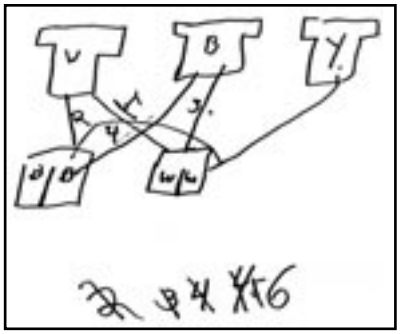
Going Further, cont'd.

Episode Box Three: Shirts and Pants in Grade 3

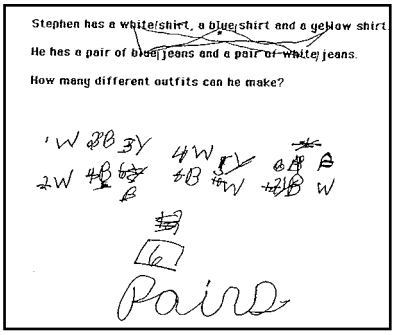
As third-graders, Dana, Stephanie, and Michael again solved the problem. There had been no intervention between the two problem sessions. This time, each child found 6 outfits as their solution.



Dana's Solution



Stephanie's Solution

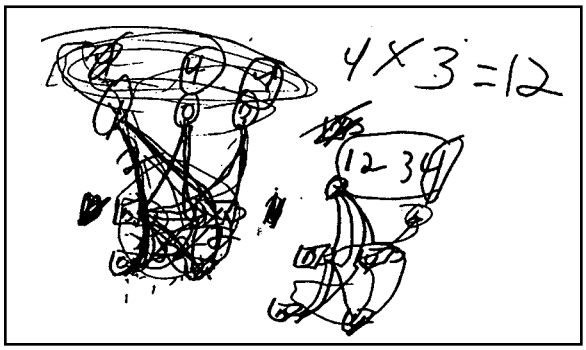


Michael's Solution

- Discuss the differences and/or similarities in the children's problem solving from grade two to grade three.

Episode Box Four: Cups, Bowls, and Plates

Stephanie and Dana solved the problem together. In a follow-up interview, Stephanie explained her solution using a diagram with lines connecting the three groups of items to find 12 combinations.



- Discuss the girls' "connecting tree" strategy.
- How might this strategy be related to the approach used in the Shirts and Pants activity?

For Next Time

Homework Assignment

Preparing for Workshop 2

1. Study the Episode Boxes on the previous pages. Reflect on the questions and record your reactions and ideas in your journal.

2. Try the Towers Four-High problem. You have two colors of Unifix® Cubes available with which to build towers. Your homework task is to make as many different-looking towers as possible, each exactly four cubes high. A tower always points up, with the little knob on top. Find a way to convince yourself and others that you have found all possible towers four cubes high and that you have no duplicates. Keep a record of your solution, including your justification, to share during Workshop 2.

After you have completed your solution for towers four cubes high, predict (without building the towers) the number of possible towers: (a) three cubes high and (b) five cubes high. Why do you think your predictions are correct?

Note: If you do not have Unifix® Cubes, cut out and use the "dark" and "light" cubes on the following pages.

3. Use any of the problems and extensions from Getting Ready of Workshop 1 with one or more of your students. Take notes of what the children do; reflect—in writing—about your own role in the activity; collect and bring all of the children's written work to the next session to share.

Reading Assignment

The reading assignment can be found in the Appendix of this guide.

Davis, R. B., Maher, C. A., and Martino, A. "Using Videotapes To Study the Construction of Mathematical Knowledge of Individual Children Working in Groups." *Journal of Science, Education and Technology,* 1(3), 177-189, 1992.

cubes

cubes

Notes

Workshop 2.
Are You Convinced?

The simple question, "Can you convince me?" is key to the mathematical success of the students in the Rutgers long-term study. This program introduces the idea of proof as one of the key ideas in mathematics. Delving into the mathematics of the Towers problem, we'll look at how two kinds of proof—proof by cases and proof by induction—naturally grow out of the need to justify and convince others. The teacher plays a critical role by using several kinds of questions to help students move toward successful justification of their answers.

Part 1—Teachers Building Proofs

Englewood, New Jersey, is a district in transition toward a more thoughtful approach to teaching and learning in mathematics. In a professional development workshop led by Arthur Powell, a researcher in the Kenilworth study, teachers in the district come up with creative and mathematically sound justifications for their solutions to the Towers problem. These solutions are similar in many ways to the solutions offered by the students, and in fact, many of the approaches arrived at independently by these teachers will appear again in the series.

On-Screen Participants: Dr. Joyce Baynes, Superintendent, Englewood Public Schools; Arthur Powell, Rutgers University; and Englewood, New Jersey Teachers, Grades K-8.

Part 2—Students Building Proofs

Returning to the Kenilworth study, we examine the research footage in detail as the students justify and convince the researchers and each other that they have found a way to determine the number of combinations that can be made in towers *n* high. That students—at this early age—can come up with mathematically sound proofs has important implications for teachers at all grade levels.

On-Screen Participants: Dr. Carolyn Maher, Rutgers University. **Student Participants:** Stephanie, Jeff, Michelle, and Milin, Kenilworth Public Schools, Grades 3 and 4.

On-Site Activities and Timeline

60 minutes

Getting Ready

1. Review

a. Discuss reflections and ideas you have had since Workshop 1.

b. Share experiences from your students' exploration of the Shirts and Pants problem.

2. Towers Four-High

a. Share individual solutions to the Towers Four-High problem from your homework in small groups and, if time permits, in the total group. As you do this, pay close attention to what appear to be important ideas, strategies, and arguments in each solution. Question each other closely about any statements or actions that are unclear.

b. After your group is convinced of its solution(s) for towers four cubes high, predict—and defend the prediction of—(a) the number of different towers three cubes high when selecting from two colors; and (b) the number of different towers five cubes high when selecting from two colors. Is it possible from your solution for towers four-high to figure out a general rule to find the number of towers of any height?

c. Is this a problem you might use with your students? How do you think that your students would solve this problem?

60 minutes

Watch the Workshop Video

Part 1

On-Screen Math Activities

Towers
Build all possible towers that are five (or four, or three, or *n*) cubes high by selecting from plastic cubes in two colors. Provide a convincing argument that all possible arrangements have been found.

Focus Question

We have seen teachers presenting a number of carefully constructed arguments for finding all of the combinations of towers four-high, when selecting from two colors. Which arguments are convincing? Why?

Part 2

On-Screen Math Activities

Towers, cont'd.
See description above.

Focus Question

What are some similarities and/or differences in the mathematical reasoning by the teachers and the students that you observed?

On-Site Activities and Timeline

30 minutes

Going Further

Please address as many of these questions as possible during your allotted time and consider the remainder as part of your homework.

Episode Box One: Teachers and Towers—Patterns and Cases

As the teachers made their presentations, Group One displayed a table with five entries to show the number of towers four cubes high in each of five sets of towers, where all the towers in a set were built using the same number of blue cubes in each tower. As they showed the drawing of the 16 towers in their solution, arranged according to these five sets (or cases), they explained that they had identified the towers in each set by "trial and error" methods.

Group Two explained that they also used cases, but different than Group One, to identify their 16 towers. Their cases were defined as sets of towers that included from one to four cubes of a color "together."

The third group also based their solution on cases; in this instance—five.

- Discuss the organization of the cases by these three groups.

Math Note: Solutions described here and in later workshops include a number of important ideas and strategies that may lead to formal mathematical proofs. Here, they include:

1. Patterns

Patterns frequently recognized and used to construct new towers and to develop a convincing solution include: (a) "opposites"—pairs of towers with the color of each cube in the first tower replaced by the alternate color to form the partner tower; (b) "flips"—pairs of towers with the color pattern from top to bottom of the first tower constructed from bottom to top to form the partner tower; (c) "staircases"—sets of towers beginning with a single bottom cube of one color in the first tower, two bottom cubes of that color in the second tower, and so on until the fourth tower is completely made up of that color; (d) "elevators"—sets of towers with one cube of the first color and three of the second, with the single-colored cube placed in each of the four possible positions.

2. Proof by Cases

The set of towers is separated into groups, or "cases," so that every tower will be included in exactly one case. A convincing argument is then made, case by case, that all possible towers for that case have been found.

On-Site Activities and Timeline

Going Further, cont'd.

Episode Box Two: Teachers and Towers—Doubling

The final presentation by the teachers was made by the group that we observed earlier in the video as they were working out a justification for their towers four-high. Their conclusions were based on solutions they had found for towers one-, two-, and three-high. This group arranged the towers of each height in a manner that showed how each new tower was built from a tower that was one cube shorter. They pointed out that for each successive height, the number of towers of that height doubled from the number found for the height one cube shorter.

- Explain why you did or did not find this to be a convincing argument.

Math Note: This solution includes ideas that lead to mathematical proof by induction. The statement is made that for towers that are one cube in height there can be only two possible towers: one of each color. From this beginning, the inductive argument is developed that if we know the number of towers n cubes high, then there must be twice that number for towers of height $n + 1$, since each of the n towers would account for two towers of height $n + 1$, built by adding one of either color to the original base tower. Tree representations may be built that show the development of this logical argument from towers one cube high to the 16 towers that are four cubes high.

Episode Box Three: Children and the Towers Problems

In the video, we saw the Kenilworth children working together in classroom sessions in grades three and four, building solutions to problems about towers. We later observed some of the same children, in interviews, working hard—even struggling and arguing with each other—to develop convincing justifications about their solutions and what they had discovered about the towers.

- What can we say about the students' methods of solving these problems and what happened to them over time?

Math Note: The mathematical term "combinatorics" is mentioned throughout the videotapes to describe particular kinds of problems. Combinatorics problems involve the mathematics of systematic counting based on strategies, such as pattern recognition and grouping. These ideas are important for children as they develop operational skills with whole numbers, basic understanding of probability and discrete mathematics, algebraic concepts such as variables, and overall ideas about justification and generalization. The Principles and Standards for School Mathematics (2000) calls for investigations involving combinatorics to be included throughout school mathematics.

For Next Time

Homework Assignment

Preparing for Workshop 3

1. Study the Episode Boxes on the previous pages. Reflect on the questions and record your reactions and ideas in your journal.

2. Try the following three Pizza problems. For each of the problems, find a way to convince yourself and others that you have found all possible pizzas and that you have no duplicates. Keep a record of how you developed each solution, including any written work that you may have done to develop your solutions and justifications, to share during Workshop 3.

After you have completed the three problems, ask yourself how they are similar to and different from each other and what similarities there may be between any of these problems and the Towers problem.

> **Pizza Problem One: Halves With Two Toppings**
> A local pizza shop has asked us to help them keep track of certain pizza sales. Their standard "plain" pizza contains cheese. On this cheese pizza, one or two toppings can be added to either half of the plain pie or the whole pie. How many possible choices for pizza do customers have if they can select from two different toppings (sausage and pepperoni) that can be placed on either the whole cheese pizza or half a cheese pizza? List all possible selections.
>
> **Pizza Problem Two: Whole Pizzas and Four Toppings**
> The pizza shop has asked for help again. This time they are offering a basic cheese pizza with tomato sauce. A customer can then select from the following toppings to add to the whole basic pizza: peppers, sausage, mushrooms, and pepperoni. How many different choices for pizza does a customer have? List all the possible selections.
>
> The local pizza shop was so pleased with our help that they have asked us to continue the work. Remember that they offer a cheese pizza with tomato sauce and that the customer can then select from four toppings: peppers, sausage, mushrooms, and pepperoni. The shop owner now offers a choice of crusts: regular (thin) or Sicilian (thick). How many choices for pizza does a customer now have? List all possible combinations.
>
> **Pizza Problem Three: Halves and Four Toppings**
> At customer request, the pizza shop has agreed to fill orders with different choices for each half of a pizza. Remember that they offer a basic cheese pizza with tomato sauce. A customer can then select from four toppings: peppers, sausage, mushrooms, and pepperoni. There is a choice of crusts: regular (thin) and Sicilian (thick). How many choices for pizza does a customer now have?

3. Use the Towers problems with your students. Observe their activity carefully and take notes on how they approach the problem, paying particular attention to how their strategies and approaches are similar to and different from the ways in which you thought about the problem and what you observed in the video.

For Next Time

Reading Assignment

The reading assignment can be found in the Appendix of this guide.

Maher, C. A. and Martino, A. M. "Conditions for Conceptual Change: From Pattern Recognition to Theory Posing." In H. Mansfield and N. H. Pateman (Eds.), *Young Children and Mathematics: Concepts and Their Representation.* Sydney, Australia: Australian Association of Mathematics Teachers, 1997.

Maher, C. A. and Martino, A. M. "Young Children Invent Methods of Proof: The Gang of Four." In P. Nesher, L. P. Steffe, P. Cobb, B. Greer, and J. Goldin (Eds.), *Theories of Mathematical Learning,* 431-447. Mahwah, NJ: Lawrence E. Erlbaum Associates, 1996.

Workshop 3.
Inventing Notations

How can a person make an idea visible or keep track of a line of thought? From kindergarten arithmetic to high school calculus, mathematics involves notations—symbols as surrogates for abstract ideas. In Workshop 3, we introduce new but mathematically related investigations: the Pizza problems, beginning with "How many different pizzas can you make by selecting from four toppings?" In this problem, students invent their own notations to represent the different toppings and the combinations that can be made with them. In Workshop 3, teachers learn how to foster and appreciate students' notations for their richness and creativity, and consider some of the possibilities that early work with notation systems might open up for students as they move toward more sophisticated math.

Part 1—Putting It on Paper: Elementary Students Invent Notations

Returning to her classroom in the fall after the summer professional development workshop depicted in Workshop 2, Englewood, New Jersey fourth-grade teacher Blanche Young tries out one of the Pizza activities with her students. Afterwards, she discusses their work with Arthur Powell, the workshop facilitator. Next, fifth-graders from another site in New Brunswick, New Jersey, develop complex and varied notation systems to organize their pizza combinations. Finally, in Colts Neck, New Jersey, a fourth-grader named Brandon comes up with a way to link the Pizza problem to the Towers problem by applying the same notation system to both.

On-Screen Participants: Blanche Young, Englewood Public Schools, Grade 4; Arthur Powell, Rutgers University; and Dr. Alice Alston, Rutgers University. **Student Participants:** Fifth-Graders, New Brunswick Public Schools; and Brandon, Colts Neck Public Schools, Grade 4.

Part 2—Notations Evolve As Students' Thinking Evolves (and Vice Versa)

In the fifth grade, students from the Kenilworth study make a series of leaps into ever more abstract representations of the Pizza problem. They build up to an extension of the problem that is so complex that they can't keep track of their combinations by counting. They find ways to extend their earlier work, inventing notations along the way, and arrive at and defend their solutions.

On-Screen Participants: Dr. Carolyn Maher, Rutgers University; and Dr. Amy Martino, Rutgers University. **Student Participants:** Stephanie, Jeff, Matt, and Michelle, Kenilworth Public Schools, Grade 5.

On-Site Activities and Timeline

60 minutes

Getting Ready

1. Review

a. Share and discuss ideas and experiences with your students concerning the Towers problems.

2. The Three Pizza Problems

In order to get the most out of the video, it is important for teachers to think about their own solutions to the problems—especially the notations, organizational schemes, and strategies that they used. You tried the three Pizza problems as homework for Workshop 2. Familiarity with each of the three problems is important in order to follow what the children in the video are doing. However, in this session, we will focus on Problem Two: Whole Pizzas and Four Toppings. For that reason, after comparing solutions to Problem One: Halves With Two Toppings, you may choose to spend the greater amount of the first hour thinking together about your approaches to Problem Two, leaving discussion of Problem Three for later.

a. Share individual solutions to each of the three problems—one at a time—in small groups and, as time permits, in the whole group. As you do this, pay close attention to what appear to be the important ideas in each solution, especially the different notations developed, the different organizations and strategies employed, and arguments used to justify solutions. Question each other closely about any statements or actions that are unclear to you.

b. After a group is convinced of its solution(s), for each problem: (a) predict—and defend your prediction of—the number of different pizza choices that would result if an additional topping was offered; and (b) ask whether this problem is similar—and in what way—to other problems that you may have encountered.

c. In what ways are the three problems different from and similar to each other?

d. Would you use any or all of these problems with your students? Why or why not? How do you think your students would solve these problems?

On-Site Activities and Timeline

60 minutes

Watch the Workshop Video

Part 1

On-Screen Math Activities

The Pizza Problem

Pizza Hut® has asked us to help design a form to keep track of certain pizza choices. They offer a cheese pizza with tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushrooms, and pepperoni. How many choices for pizza does a customer have? List all the possible choices. Find a way to convince each other that you have accounted for all possibilities.

Focus Question

What do these three examples—Englewood, New Brunswick, and Colts Neck—have in common in terms of how the notations help students justify their solutions?

Part 2

On-Screen Math Activities

The Pizza Problem With Halves

Capri Pizza has asked us to help design a form to keep track of certain pizza sales. Their standard "plain" pizza contains cheese. On this cheese pizza, one or two toppings can be added to either half of the plain pie or whole pie. How many choices do customers have if they can choose from two different toppings (sausage and pepperoni) that can be placed on either a whole cheese pizza or half of a cheese pizza? List all possibilities. Show your plan for determining these choices. Convince us that you have accounted for all possibilities and that there could be no more.

Focus Question

We've seen students spontaneously creating ways of keeping track of their solutions to a problem. What notations are students using to represent their ideas and organize the pizzas?

On-Site Activities and Timeline

30 minutes

Going Further

Please address as many of these questions as possible during your allotted time and consider the remainder as part of your homework.

Episode Box One: Exploring Pizza Choices in Classroom Contexts

In the classrooms in Englewood and New Brunswick, we observed children working together in groups on Problem Two: Whole Pizzas and Four Toppings. They developed various ways to represent and keep track of pizza choices and then justify their solutions to others. We also listened to Blanche, the fourth-grade Englewood teacher, as she discussed her concerns for her students.

- What notations and strategies for keeping track of pizza selections did the children use?
- What arguments did the children use to justify their solutions?

Episode Box Two: Making Connections—Brandon

We saw fourth-grader Brandon as he worked on Problem Two with his partner, explained his notation and approach to the problem in an interview, and then connected this problem to Towers Four-High. His initial ideas about connections between the two problems involved surface characteristics. However, as he reorganized his towers into cases and began to recognize the structural identity between the two problems, Brandon mapped the 16 towers in a one-to-one relationship to the 16 pizza selections, based on his "zero-one" notation.

- What mathematical ideas were involved in Brandon's solutions?

On-Site Activities and Timeline

Going Further, cont'd.

Episode Box Three: Developing Mathematical Ideas With Pizza Variations

As fifth-graders, the Kenilworth students worked on the series of three pizza problems during two extended classroom sessions approximately one week apart, in the same order they were assigned to you. When solving Problem One in the first session, the students worked in two groups. Each group developed particular notations and grouped the pizza choices by categories, or cases, in order to accurately account for the 10 possible selections.

- What do you notice about the children's problem solving?

In the second session the students first revisited their solutions for Problem One, sharing strategies and ideas. They were then asked to solve Problem Two. Working singly and in small groups, the students developed various notations to keep track of choices and solved the problem quickly, grouping their selections by cases that categorized the pizzas as having: (1) no extra toppings, (2) exactly one topping, (3) two toppings, (4) three toppings, and (5) all four toppings. The extension to two crusts was solved immediately by doubling the 16 pizza choices that they had found. Problem Three proved to be extremely challenging to the students until Matt developed a notation and argument based on the original 16 "whole pizza selections" for Problem Two. He categorized possibilities for pizzas with different halves into cases by holding constant each of the original 16 selections in turn. For each case, the selection held constant became one half of the pizza, while the second half was composed of each of the choices that followed. Matt's final solution was to sum the numbers for each case (15 + 14 + 13 + ... + 2 + 1 = 120) and then add the original 16 whole pizzas (136 possible choices). He credited his approach to Ankur's strategy for Problem One.

- Did you find Matt's argument convincing? Why or why not?

For Next Time

Homework Assignment

Preparing for Workshop 4

1. The assignment for Workshop 4 includes two problems relating to different kinds of towers. These problems will be central to the video in Workshop 4. The first is the classic Towers of Hanoi problem and the second involves Towers Four-High built from Unifix® cubes, this time selecting from three colors.

For each of the two problems, find a way to convince yourself and others that your solution is correct. Keep a record of how you developed and justified each solution, including any written work that you may have done, to share during Workshop 4. After you have completed the problems, ask yourself how they are similar to and different from each other, and what similarities there might be between these problems and other problems that you have encountered.

> **Problem One: Towers of Hanoi**
> Legend has it that a group of Eastern monks are the keepers of three towers on which sit 100 golden rings. Originally, all 100 rings were stacked on one tower with each ring smaller than the one beneath. The monks' task was to move the entire stack of rings from this first tower to the third tower one at a time, using the second tower as necessary, but never placing a larger ring on top of a smaller one. The legend was that once all 100 rings have been successfully moved, the world will come to an end.
>
> Dr. Robert B. Davis presented this problem to the Kenilworth sixth-graders in the following way:
>
>> "There is an order of monks in the city of Hanoi and they were concerned about when the world is going to end. So they made a puzzle like this."
>>
>> Dr. Davis held up a Towers of Hanoi model—a rectangular base with three dowels attached perpendicular to that base. Eight rings were stacked on one of the dowels, with each of the eight smaller than the one immediately below it.

For Next Time

Homework Assignment, cont'd.

> "The monks spend all of their time working to solve the puzzle. When they have it done, that is supposed to be when the world ends. I thought it might be interesting to figure out when the world is going to end, so that we would know it, too. So the question is this: How many 'moves' (the smallest possible number) would it take to complete the task?
>
> "Let's agree on what the rules are: (1) You can move exactly one disk at a time. (2) You can never put a bigger disk on top of a smaller one.
>
> "So how many 'moves' would it take to complete the task?"

Problem Two: Ankur's Challenge With Towers Four-High
In the 10th grade, Ankur presented the following problem to the group:

> Find as many towers as possible that are four cubes high if you can select from three colors and there must be at least one of each color in each tower. Build a solution for this problem, selecting from three colors of Unifix® cubes.
>
> How do you know that you have found all the possibilities? Convince your peers that you have found all the possibilities—no more and no fewer.

2. Use the Pizza problems with your students. Observe their activity carefully and take notes about how they approach the problem, paying particular attention to how their notations, strategies, and approaches are similar to and different from the ways that you thought about the problem and what you observed in the video. Collect interesting samples of notation from your students for your journal and to share in Workshop 4.

Reading Assignment

The reading assignment can be found in the Appendix of this guide.

Maher, C. A. "The Nature of Learning." In *Can Teachers Help Children Make Convincing Arguments? A Glimpse Into the Process,* 21-34. Universidade Santa Ursula: Rio de Janeiro, Brazil, 1998.

Maher, C. A., and Martino, A. M. "Brandon's Proof and Isomorphism." In *Can Teachers Help Children Make Convincing Arguments? A Glimpse Into the Process,* 77-101. Universidade Santa Ursula: Rio de Janeiro, Brazil, 1998.

Notes

Workshop 4.
Thinking Like a Mathematician

Towers of Hanoi is a game played by Buddhist monks. According to the rules of the game, when the last move of the game is performed, the world will end. In the sixth grade, students in the Kenilworth study try this game and look for a general formula to find the total number of moves. The steps they follow in solving the "mystery" closely parallel some strategies that professional mathematician Fern Hunt uses in her work.

Part 1—Strategies for Solving Problems

Our brief profile of mathematician Fern Hunt shows that collaboration is an essential part of doing mathematics in the real world. Fern uses the metaphor of a game (The Towers of Hanoi) to show the types of strategies that mathematicians might use in solving real-world problems. Under the guidance of the late Robert B. Davis, Kenilworth sixth-graders put into practice some of these strategies, working together enthusiastically on the challenge of uncovering the mystery of the Towers of Hanoi.

On-Screen Participants: Dr. Fern Hunt, National Institute for Science and Technology; and Dr. Robert B. Davis, Rutgers University. **Student Participants:** Brian, Jeff, Romina, Michael, Matt, Stephanie, Amy Lynn, Dana, Ankur, and other sixth-graders, Kenilworth Public Schools.

Part 2—Encouraging Students To Think

In Provo, Utah, Presidential-award-winning math teacher Janet Walter has just inherited a ninth-grade Algebra I class mid-year. She is trying to help her students be more confident discussing their developing mathematical ideas. In Kenilworth, Mike, a 10th-grade student in the long-term study, shares his insights. His ideas quickly evolve into a firmly established methodology that is used again and again to solve problems. Romina, another student in the study, responds to a challenge presented by Ankur, one of her classmates, and presents her solution to the group.

On-Screen Participants: Janet Walter, Provo, Utah Public Schools. **Student Participants:** Algebra I Class, Provo, Utah; Michael, Jeff, Romina, Ankur, and Brian, Kenilworth Public Schools.

On-Site Activities and Timeline

60 minutes

Getting Ready

Share and discuss solutions to Ankur's Challenge and Towers of Hanoi, your homework from Workshop 3.

60 minutes

Watch the Workshop Video

Part 1

On-Screen Math Activities

Towers of Hanoi
See pages 34-35.

Focus Question

We've seen students using a variety of problem-solving strategies to approach the Towers of Hanoi problem. What strategies have you observed your students using to solve difficult problems?

Part 2

On-Screen Math Activities

Scatter Plot (Provo, Utah)
The students' assignment was to plot a number of data points on a coordinate grid and then draw a line that best describes the data. They followed up by writing an equation for the line in the form: $y = ax + b$.

10th Grade: Revisiting the Pizza Problem
How many pizzas can you make by selecting from four different toppings?

Romina's Proof
How many different towers can you make four blocks high when selecting from three colors, if there is at least one of each color in each tower?

Focus Question

Is Romina's argument convincing? Why or why not?

On-Site Activities and Timeline

30 minutes

Going Further

Please address as many of these questions as possible during your allotted time and consider the remainder as part of your homework.

Episode Box One: Fern Hunt—Mathematician

Fern Hunt tells us her view on how mathematicians solve problems. For example, she points out that principles found in games and puzzles often engage the creative powers of the mind.

- Discuss this idea.

She indicated further that mathematicians, in dealing with complex problems, often simplify the situation in order to look at a related, but simpler, problem.

- How do you relate Fern Hunt's ideas to how you solve problems? To how the students you teach solve problems?

On-Site Activities and Timeline

Going Further, cont'd.

Episode Box Two: Sixth Grade—The Towers of Hanoi

Mathematician Fern Hunt used the Towers of Hanoi problem as an example of how she and other mathematicians solve problems. In the sixth grade, the students in the Kenilworth study worked on the same problem.

- Relate Fern Hunt's ideas about how mathematicians solve problems to the way the sixth-grade children work on the Towers of Hanoi problem in the video.

Michael explains the "doubling plus one" rule by moving stacks of discs of a particular height.

- Explain his idea. Is it a convincing argument for the rule? Why or why not?

The sixth-graders in this video clip are posing rules for explaining the Towers of Hanoi puzzle for stacks of discs *n* high. Matt suggests that algebra is being used when he offers that "Two *w* plus one equals the number you say."

- What evidence, if any, do you see of algebra or algebraic thinking in the explanations and work of these students? What is your view of the challenge given to these students for finding a general solution? What mathematical ideas are the students using in the pursuit of the solution?

- Comment on the table produced by the students for recording their data.

□	△	
1	1	
		2
2	3	
		4
3	7	
		8
4	15	
		16
5	31	
		32
6	63	
		64
7	127	

On-Site Activities and Timeline

Going Further, cont'd.

Episode Box Three: Ninth Grade—Algebraic Thinking

Janet was assigned a ninth-grade Algebra I class part-way through the year.

- Discuss Janet's way of working with the students. How engaged are they? What mathematical ideas are the students building?

Episode Box Four: 10th Grade—Michael's Binary Code

Michael is working with a group of students in an after-school group interview session.

- What mathematics is Michael doing?

Recall that Brandon recorded his solution to the four-topping Pizza problem with 0's and 1's and related this work to the Towers Four-High problem.

For example, Brandon, grade four, wrote:

1000
0100
0010 etc.

Recall, Michael's notation in grade 11:

0001
0010
0011 etc.

- How does Michael's binary code differ from Brandon's? Are there advantages or disadvantages to each system? Discuss the use of notation in the problem solving of these students.

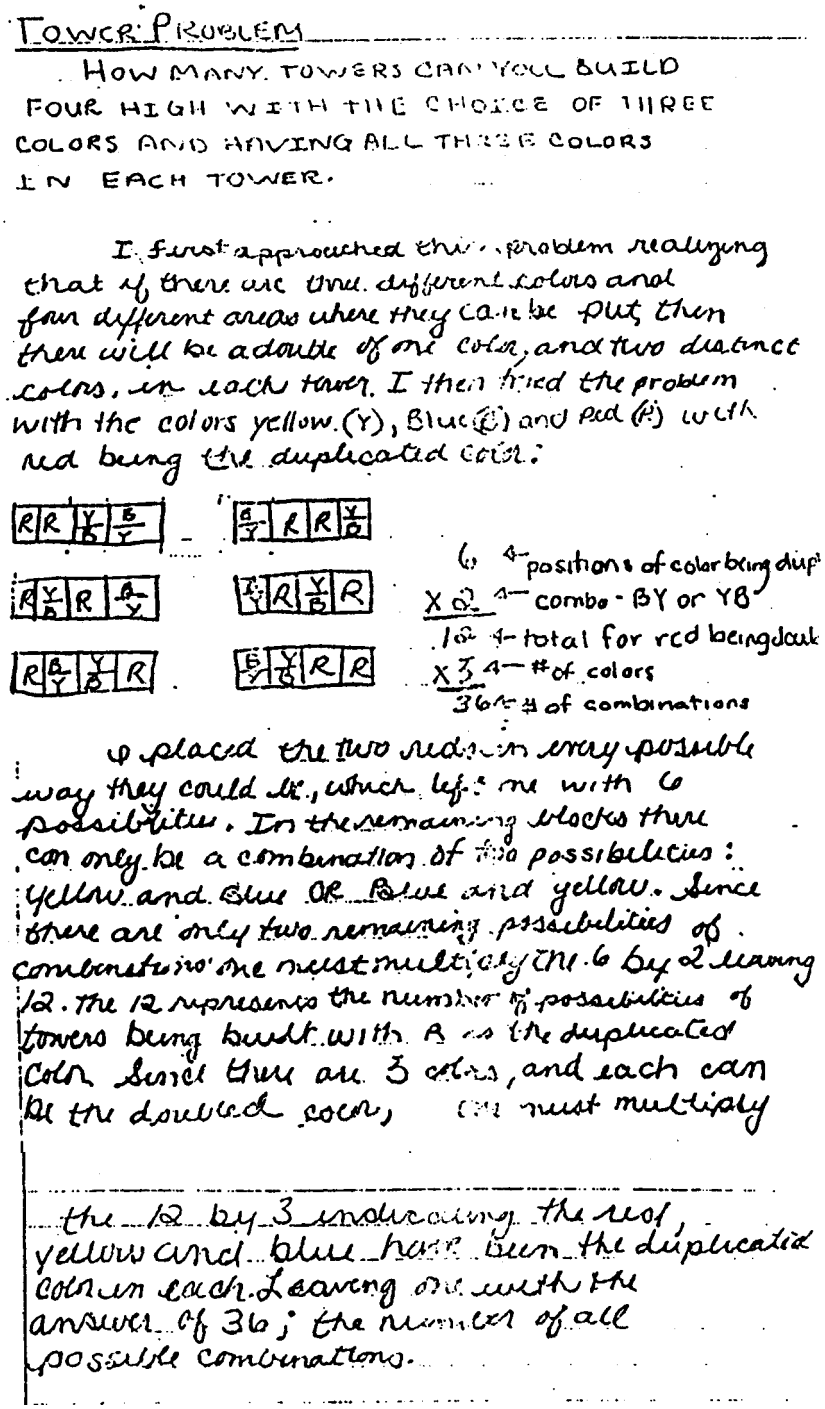
Episode Box Five: 10th Grade—Romina's Proof

Ankur presented the following problem to the group: "Find as many towers as possible that are four cubes tall if you can select from three colors and there must be at least one of each color in each tower. How do you know that you have found all the possibilities? Build a solution, selecting from three colors of Unifix® cubes. Convince your peers that you have found all the possibilities, no more and no fewer."

On the next page is Romina's solution to Ankur's challenge.

- Discuss Romina's solution and its presentation to the group. Were you convinced? Why or why not?

Going Further, cont'd.



TOWER PROBLEM

HOW MANY TOWERS CAN YOU BUILD FOUR HIGH WITH THE CHOICE OF THREE COLORS AND HAVING ALL THREE COLORS IN EACH TOWER.

I first approached this problem realizing that if there are three different colors and four different areas where they can be put, then there will be a double of one color, and two distinct colors, in each tower. I then tried the problem with the colors yellow (Y), Blue (B) and Red (R) with red being the duplicated color:

R R Y/B B/Y B/Y R R Y/B

R Y/B R B/Y B/Y R Y/B R

R B/Y Y/B R B/Y Y/B R R

6 ← positions of color being dup
X 2 ← combo - BY or YB
12 ← total for red being dou
X 3 ← # of colors
36 ← # of combinations

I placed the two reds in every possible way they could be, which left me with 6 possibilities. In the remaining blocks there can only be a combination of two possibilities: yellow and blue OR blue and yellow. Since there are only two remaining possibilities of combination, one must multiply the 6 by 2 leaving 12. The 12 represents the number of possibilities of towers being built with R as the duplicated color. Since there are 3 colors, and each can be the doubled color, one must multiply the 12 by 3 indicating the red, yellow and blue have been the duplicated color in each leaving one with the answer of 36; the number of all possible combinations.

Romina's Solution

For Next Time

Homework Assignment

Preparing for Workshop 5

1. Study the Episode Boxes on the previous pages. Reflect on the questions and record your reactions and ideas in your journal.

2. Solve the World Series problem. In a World Series, two teams play each other in at least four and at most seven games. The first team to win four games is the winner of the World Series. Assuming that both teams are equally matched, what is the probability that a World Series will be won (a) in four games? (b) in five games? (c) in six games? (d) in seven games?

Notes

Workshop 5.
Building on Useful Ideas

One of the most important conditions of the Rutgers' long-term study was that students were invited to work together to solve problems. By sharing and justifying their ideas, students are able to clarify their own thinking. Collaborative work thus becomes the vehicle for advancing each individual student's ideas. Workshop 5 focuses on how teachers can foster thoughtful mathematics through building a learning community.

Part 1—The Changing Role of the Teacher: Elementary Classrooms

Three teachers who participated in the Englewood summer professional development workshop try activities in their elementary school classrooms. The teachers include Michelle, a kindergarten teacher; Melissa, a second-grade teacher; and Blanche, a fourth-grade teacher. We observe Amy, a researcher in the long-term study, as she presents an activity to her fourth-grade students in Colts Necks, New Jersey. Please note the mathematical activities, how these activities are introduced by the teachers, and what strategies their students employ in solving the tasks.

On-Screen Participants: Blanche Young, Englewood Public Schools, Grade 4; Melissa Sharpe, Englewood Public Schools, Grade 2; Michelle Doherty, Englewood Public Schools, Kindergarten; and Dr. Amy Martino, Colts Neck Public Schools.

Part 2—Pascal's Triangle and High School Algebra

As a follow-up to earlier Pizza and Towers tasks, high school math teacher Dr. Gina Kiczek invites her Jersey City, New Jersey, students to solve other counting problems. In another setting, Stephanie, a student in the long-term study, connects the Towers activity to Pascal's Triangle in grade 8, and then the Pizza and Towers problems to Pascal's Triangle in grade 11. Then observe a group of 11th-graders from the long-term study as they investigate the World Series problem, and later link Pascal's Triangle to standard notation in the form of "*n* choose *r*."

On-Screen Participants: Dr. Regina Kiczek, James J. Ferris High School, Jersey City, New Jersey. **Student Participants:** Stephanie, Michael, Romina, Brian, Jeff, and Ankur, Kenilworth Public Schools.

On-Site Activities and Timeline

60 minutes

Getting Ready

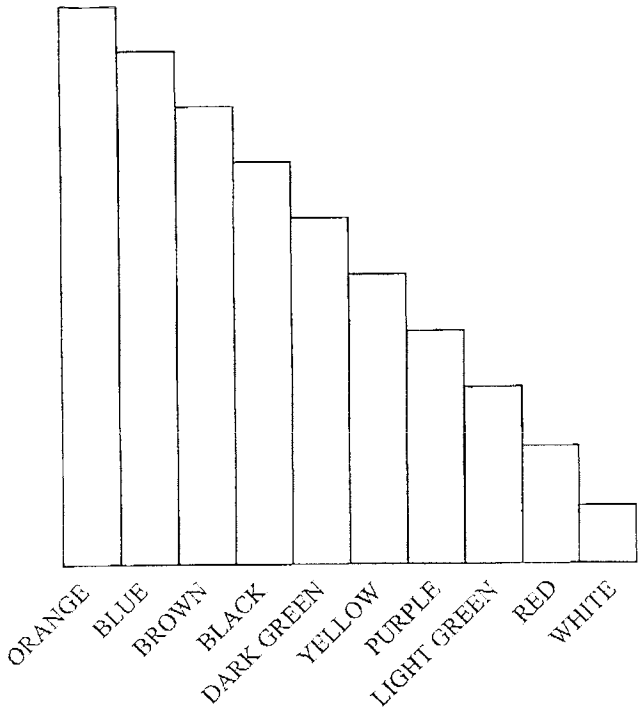
1. The World Series Problem

Your homework for today's workshop was the World Series problem. Share your results in small groups.

Discuss as many of the following as time will allow:

2. The Cuisenaire® Rod Activities

Cuisenaire® Rods are used in the kindergarten and second-grade activities as shown in the video. The traditional set of rods that students use is designed in 1cm increments, starting with white as 1cm. The rod lengths in the set used by the kindergarten teacher and her students are proportionally larger than the traditional set of rods. There are 10 rods in each set; each rod has a permanent color name but has deliberately not been given a permanent number name. For example, the length of the dark green rod might be called "four" in one activity and "one" in another.



Cuisenaire® Rods referenced here by permission from ETA/Cuisenaire® , Vernon Hills, Illinois. All rights reserved.

On-Site Activities and Timeline

Getting Ready, cont'd.

2. The Cuisenaire® Rod Activities, cont'd.

"Trains" can be constructed by placing rods together. Trains may be multiples of the same rod, or a mix of different rods. The children construct trains to aid them in finding solutions to the given problems.

> How many different ways can we make dark green? (Kindergarten)
>
> What are all the different ways that we can make a train equal to the length of one magenta rod? (Second Grade) Please note: Cuisenaire® refers to this rod as "purple."
>
> a. Is this a problem you might use with your students? How do you think your students would solve this problem?
>
> b. An extension to the activities: Can you determine how to find the answer to a similar question for a rod of any length?

3. The Ice Cream Problems

In today's video, math teacher Gina Kiczek presents the problem below to her high-school students. What mathematical ideas do you believe she wants them to build?

> The new pizza shop in the Heights has been doing a lot of business. The owner thinks that it has been so hot this season that he would like to open up an ice cream shop next door. Because he plans to start out with a small freezer, he decides to initially sell only six flavors: vanilla, chocolate, pistachio, boysenberry, cherry, and butter pecan.
>
> Bowls: The cones that were ordered did not arrive in time for the grand opening so all the ice cream was served in bowls. How many choices for bowls of ice cream does the customer have? Find a way to convince each other that you have accounted for all possibilities.
>
> Cones: The cones were delivered later in the week. The owner soon discovered that people are particular about the order in which the scoops are stacked. "After all," one customer said, "eating chocolate then vanilla is a different taste than eating vanilla then chocolate." The owner also discovered rather quickly that he couldn't stack more than four scoops in a cone. How many choices for ice cream cones does a customer have? Find a way to convince each other that you have accounted for all possibilities.

4. Discuss Pascal's Triangle

Refer to the Pascal's Triangle Worksheet on the following pages.

a. Can you model Pascal's Triangle with block towers?

b. How does the doubling rule work?

c. Can you explain how and/or why the addition rule works?

Pascal's Triangle Worksheet

Patterns provide much of the backbone and motivation in mathematical problem solving. This is true for children as well as for professional mathematicians.

In the video you will see Stephanie's exploration of patterns leading to a comparison between Pascal's Triangle and the Towers Problem.

In the early grades, children were seen on video establishing such patterns to help them determine how many different towers there are and to try to convince themselves and others when they believe they have found all of the possibilities. In the video clips of the children in the early grades you can see several patterns emerging:

note: n = the height of the tower

blue	0	1	2	3	. . .	n
2nd color	n	n-1	n-2	n-3	. . .	0
towers 1-tall	1					1
towers 2-tall	1	2				1
towers 3-tall	1	3	3			1
towers 4-tall	1	4	6	4		1

.
.
.

Sometimes the numbers above are written in triangular arrangement:

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

.
.
.

- The mathematics involved in Pascal's triangle forms an important starting point for the branch of mathematics known as combinatorics. Find the pattern represented in the triangle.

The Summing Relationship:
Given that we know a row in Pascal's Triangle, the entries in the next row can be found in the following manner:

The first number in the new row will be 1.

The second number in the new row will be the sum of the 1st and 2nd numbers in the previous row.

The third number in the new row will be the sum of the 2nd and 3rd numbers in the previous row.

.
.
.

And finally the last number in the new row will be 1.

Another relationship among the numbers in Pascal's triangle fits with the children's earlier discovery that as the height of the towers increase by 1 block, the number of different possible towers doubles. If you sum the numbers in any row of Pascal's Triangle, you will observe that those sums double as you progress down the rows.

1 + 1	= 2	$= 2^1$
1 + 2 + 1	= 2 x 2	$= 2^2$
1 + 3 + 3 + 1	= 2 x 2 x 2	$= 2^3$
1 + 4 + 6 + 4 + 1	= 2 x 2 x 2 x 2	$= 2^4$

etc.

The mathematics involved in Pascal's triangle now forms an important starting point for the branch of mathematics known as combinatorics.

Questions:

1. Can you model Pascal's triangle with block towers?

2. How and why does the doubling rule work?

3. Can you explain how and why the addition rule works?

On-Site Activities and Timeline

60 minutes

Watch the Workshop Video

Part 1

On-Screen Math Activities

Trains (Kindergarten)
Students arrange shorter rods end-to-end to match the length of a given longer rod.

Trains (Second Grade)
Students try to find all possible ways to arrange shorter rods end-to-end to match the length of a given rod. They count the number of possibilities and compare results.

Towers (Fourth Grade)
Students try to find out how many different towers four blocks high they can build by selecting from blocks of two colors.

Focus Question

What actions taken by these teachers across the grade levels seem to encourage students to think mathematically? In what way are these effective?

Part 2

On-Screen Math Activities

Ice Cream Problems
Bowls: There are six flavors of ice cream. If the ice cream is served in bowls that can hold up to six scoops, how many different ways can the ice cream be served?

Cones: In a variation of the problem, the ice cream can be served in cones stacked up to four scoops high. Given that the order of stacking matters, how many different cones could be served?

Building Pascal's Triangle
A researcher (Robert Speiser) probes Stephanie's understanding of the relationship of the numbers in row n of Pascal's Triangle to towers n high when choosing from two colors.

World Series Problem
Two evenly matched teams play a series of games in which the first team to win four games wins the series. What is the probability that the series will be decided in 1) four games? 2) five games? 3) six games? or 4) seven games?

"*n* choose *r*"
Students derive the formula for determining the number of ways that a subset of r objects can be selected from a total of n objects.

Focus Question

How do the Pizza problems, Towers problems, and World Series problem relate to Pascal's Triangle?

On-Site Activities and Timeline

30 minutes

Going Further

Please address as many of these questions as possible during your allotted time and consider the remainder as part of your homework.

Episode Box One: Englewood—Rods in Kindergarten

In the fall, a kindergarten teacher who attended the Englewood summer workshop was able to send half of her students to a specialist while she gathered the rest (approximately 12) in a circle. She challenged them to propose ways of stacking shorter rods end-to-end to match a given rod. Each student's proposal was considered by the group.

The children offered the following trains as solutions to the problem "How many different ways can we make dark green"?: 2 light green rods; 1 white rod and 1 yellow rod; 3 red rods; and 6 white rods. Jacob suggested a train of 2 white rods and 1 red rod. After a discussion led by the teacher, Jacob then offered 1 red rod, 2 white rods, and 1 red rod as his model. Other children then suggested a train of 2 white rods and 1 purple rod, and a train of 3 white rods and 1 light green rod.

- What mathematical ideas are the children learning in their comparisons of the rods?

The teacher commented on her enthusiasm for learning mathematics with her students.

- Comment on her views.

Episode Box Two: Englewood—Rods in Second Grade

In a second-grade class, Melissa, a first-year teacher, has been working with trains—short lengths of Cuisenaire® Rods placed end-to-end. She challenges the students to find all the different ways (order doesn't count) of combining shorter rods to match a longer rod of a given length.

We observe different children working in small groups to find possible solutions. One girl has found eight solutions and her partner has found six. The teacher asks, "Why do you think you have them all? How do you know?"

- List some benefits you think this kind of questioning might afford the students as well as the teacher's assessment of student understanding.

Notice that the teacher discusses the use of letters to keep track of the trains the students build.

- What are the advantages and disadvantages of the teacher's suggesting a form of notation for use by the children?

On-Site Activities and Timeline

Going Further, cont'd.

Episode Box Three: Englewood—Towers in Fourth Grade

The voiceover tells us that the teacher introduces the Towers activity by asking her students to make as many towers as they can by selecting from blocks of two colors. This is the first time that this teacher has tried the Towers activity with her students, who will work in pairs or small groups. The teacher states, "Make as many four-tall towers as you can, but when you're done, have as many different arrangements as you can." The activity is listed on the blackboard as "Problem Solving." It is written: Construct as many four-tall towers that are different.

- Have you heard different language in the presentations of this problem? Does it matter how the problem is stated? Discuss.

Episode Box Four: Colts Neck—The Equation Problem in Fourth Grade

Amy, the teacher in this activity, spent several years as a researcher at Rutgers during the Kenilworth study—first as a doctoral student and then as a member of the staff. She has been back in the classroom for approximately six years. A typical day begins when her fourth- graders file into the room and start working quietly on a problem she has placed on the blackboard. Today's problem is to write as many different equations as they can in which one side of the equation is 10.

- What mathematics are the children doing in this activity?

As the children shared their results, one boy, Brian, shared the following: 1000 + 1000 – 2000 + 5000 + 5000 – 10000 + 10. The teacher asked, "Do you think it makes a difference if we go from left to right… or right to left? Do we come up with the same 10?"

- Discuss the teacher's question.

On-Site Activities and Timeline

Going Further, cont'd.

Episode Box Five: Jersey City— The Ice Cream Problems in 10th Grade

Gina is nearing the end of a unit on combinatorics. Her students spent several days doing combinations problems, including Towers and Pizzas, and relating these to Pascal's Triangle. She has now introduced a new task: "How many different bowls of ice cream with up to six scoops can be made by selecting scoops from up to six different flavors?"

- What mathematical ideas do these problems raise? How were students dealing with the idea of order in these problems? Suggest follow-up activities.

Episode Box Six: Kenilworth—A Student Connects Towers to Pascal's Triangle

In an eighth-grade interview, Stephanie drew a series of comparisons between her previous work with Towers and Pascal's Triangle. She showed how each number in a row of Pascal's Triangle represents the number of towers in each subset of towers, organized by the number of blocks of a given color. She also demonstrated the "addition property," showing why each number in a new row (except the second row) of Pascal's Triangle is the sum of the two numbers directly above it.

- How did Stephanie build on her earlier ideas with towers and pizzas to explain how the addition property makes sense?

On-Site Activities and Timeline

Going Further, cont'd.

Episode Box Seven: Kenilworth—The World Series Problem in 11th Grade

Students from the Kenilworth focus group spent a two-hour after-school session working on the World Series probability problem: What are the probabilities for ending a series between two evenly matched teams in four, five, six, or seven games respectively, if the first team to win four games wins the series?

- Describe Jeff's explanation for winning a four-game world series. Why do you think Jeff felt he did not give a convincing argument for winning the series in five games? What did Michael contribute? How was Pascal's Triangle related to Michael's justification? Discuss why you were or were not convinced by the students' solutions. Why do you think Pascal's Triangle keeps coming up?

For Next Time

Homework Assignment

Preparing for Workshop 6

1. Study the Episode Boxes on the previous pages. Reflect on the questions and record your reactions and ideas in your journal.

2. Piece together the two parts of the Catwalk photographs on the following pages and photocopy them onto 11 x 17 paper. This series of 24 photographs shows a cat first walking, then running. The interval between successive frames is .031 seconds. The cat is moving in front of a grid whose lines are five centimeters apart. Some lines are darker than others. Based on these photographs, answer as best you can the following two questions.

a. How fast is the cat moving in frame 10?

b. How fast is the cat moving in frame 20?

These photographs were first published in 1885. They give the only information we have about the cat. Please base your responses on information you can gather from the photos and explain carefully how you arrived at your conclusions.

Reminder: For the next workshop, each participant will need two copies of the Catwalk photographs on 11 x 17 paper and on transparencies; metric rulers (clear plastic ones work best); graph paper; a calculator (graphing calculator if possible); and pens or markers for preparing solutions to the problems. You will also want to have an overhead projector, blank transparencies, and pens for participants to use for sharing solutions.

3. Use one of the tasks from Workshop 5 with your students. Carefully observe their activity and take notes about how they approach the problem, paying particular attention to how their strategies and approaches are similar to and different from the ways that you thought about the problem and to what you observed in the video.

Reading Assignment

The reading assignment can be found in the Appendix of this guide.

Maher, C. A. and Speiser, M. "How Far Can You Go With Block Towers? Stephanie's Intellectual Development." *The Journal of Mathematical Behavior,* 16(2), 125-132, 1997.

Join Here

+

cats

Join Here

+

cats

Notes

Workshop 6.
Possibilities of Real-Life Problems

This session builds on student work from July 1999, part of a two-week Summer Institute for high school students in Kenilworth, New Jersey. The students here include several students seen in prior workshops. Four days toward the end of the Institute, they were devoted to a single task called the Catwalk, in which students are invited to work closely with a set of photos of a moving cat. At Kenilworth, Catwalk was tried for the first time with high school students who had never taken a calculus class.

Part 1—The Catwalk: Representing What You Know

Researchers introduce the cat problem—attempting to measure the instantaneous speed of a cat in photographs taken by Eadweard Muybridge, around 1880. The students pounce on the problem—literally—as they run like a cat along a representation of the cat's journey recreated at a human scale.

On-Screen Participants: Prof. Charles Walter, Brigham Young University; and Prof. Robert Speiser, Brigham Young University. **Student Participants:** Former and present participants in the Rutgers long-term study from Kenilworth, New Jersey and New Brunswick, New Jersey Public Schools, Grade 12.

Part 2—Betting on What You Know

In real-life questions, numerical "answers" need to be interpreted in terms of the problem to be solved. What conclusions can we draw from what we found? Here, to build interpretations that will make sense of the numbers they have calculated, students focus on building and relating different representations of the information the photos make available, and connect these to their own experience. The program ends with the 2000 Kenilworth high school graduation. Students who participated in the Rutgers long-term study ponder: Does the way you study math make a difference?

Student Participants: Students from Kenilworth Public Schools, Grade 12; and New Brunswick Public Schools.

On-Site Activities and Timeline

60 minutes

Getting Ready

1. Catwalk

Last session's homework was to examine the Catwalk photographs and decide:

a. How fast is the cat moving in frame 10?

b. How fast is the cat moving in frame 20?

In small groups, discuss your ideas about the Catwalk. Be prepared to explain to the class (a) what you found, (b) what difficulties you may have encountered, (c) key choices that you made along the way, and (d) remaining questions you may wish to pursue.

Is this a problem you might use with your students? How do you think your students might solve this problem?

60 minutes

Watch the Workshop Video

Part 1

On-Screen Math Activities

Catwalk

Students are given a series of photographs of a cat moving from left to right against a grid background. The photos were taken a little less than 1/30 of a second apart. Students measure the movement of the cat against the grid and try to calculate its instantaneous speed in frames 10 and 20.

Focus Question

How might the personal experience of running help these students to deepen, organize, or clarify their growing understanding of the motion of the cat?

Part 2

On-Screen Math Activities

Catwalk, cont'd.

See description above. Students discuss the level of certainty they have about the accuracy of their answers.

Focus Question

What conditions in the mathematics classroom might be required in order to make mathematics a meaningful subject for all student all the way through high school?

On-Site Activities and Timeline

30 minutes

Going Further

Please address as many of these questions as possible during your allotted time and consider the remainder as part of your final assignments.

Episode Box One: Aquisha Uses Pairs of Transparencies

Romina explains to Chuck how she, Magda, and Aquisha are working frame-by-frame, using transparencies of the cat photos. Aquisha has developed a method of measuring where she places one transparency on top of another. We see Aquisha pass a pair of transparencies, one placed carefully on top of the other, to Magda. We watch as Magda studies these transparencies.

- What, precisely, did Aquisha do with her transparencies? What advantage do you think she gained? What distances did these students measure on the photos? How does distance measured on the photos correspond to the distance traveled by the cat?

Episode Box Two: Shelly

Shelly reports that her group has found three different answers at Frame 10. They don't yet know if what they've done is right.

- What do you think about the situation in Frame 10? On what basis might you say that you are right?

Episode Box Three: Angela and Shelly at the Overhead

Angela (at overhead) and Shelly (foreground) present their findings for Frame 10. They obtain quite different average velocities for Frames 9-10 and Frames 10-11. They average these velocities, and they propose this average as the cat's velocity at Frame 10.

- How do their average velocities (for Frames 9-10, and for Frames 10-11) compare with yours? What do you think about averaging such different results?
- Is the situation at Frame 10 significantly different from the situation in Frame 20?

On-Site Activities and Timeline

Going Further, cont'd.

Episode Box Four: Aquisha's Line Representation

Aquisha goes to the overhead to explain how she found the distances the cat had traveled between frames. She marked these distances as 23 separate line segments, and then assembled these into a composite picture of the cat's movement through all 24 frames.

- How does Aquisha obtain her 23 line segments from the photos?
- How does she put them together, to construct the line representation?
- How might this line representation help us to make sense of the cat's motion at Frame 10? What does this representation help us see?
- How did others in the room respond?

Episode Box Five: Matt Discusses Acceleration

In Part 1 of the video, Mike displayed a graph of position versus time and discussed velocities. Here, with Romina's graph of velocity versus time displayed, Matt discusses acceleration.

- What is similar and what is different between these two discussions? In particular, how are position, velocity, and acceleration being represented?
- How are these representations being used?

Episode Box Six: Romina's Question

Despite his moral and legal objections to betting, Victor just proposed exactly the same velocity for Frame 10 (namely, 145.161 cm/sec) that Angela and Shelly had presented earlier, with exactly the same reasoning. Romina questions Victor's rationale.

- When, in your experience, has averaging made sense? Why?
- In posing her question, Romina stresses the "big jump" between the interval velocities (2/.031 and 7/.031 cm/sec) which Victor and the others averaged. Why might this "jump" in speed be important here?

On-Site Activities and Timeline

Going Further, cont'd.

Episode Box Seven: Matt

Matt gives Mr. Pantozzi a criterion for when he ought to bet. Let's reflect on the students' discussions, over the whole tape, that led up to this criterion.

- How many different ideas contribute necessary evidence for what Matt is suggesting?
- How many different people made important contributions in the process?

Preparing for the Future

Final Assignments

1. Study the Episode Boxes on the previous pages. Reflect on the questions and record your reactions and ideas in your journal.

2. Now study your journal. We have looked at learning, starting with young children, following the story of their growing mathematical understanding over more than 10 years. How might their early building be reflected in the work they did on Catwalk? Is it a direct application of ideas built in the early grades? Is it in the way they reason and demand careful explanation? Is it in the way they rethink, reformulate, and then rebuild what they have done before? Is it in the way they listen to each other?

3. Try the Catwalk, or another real-world task involving motion with your students. Take notes of what the learners do; reflect—in writing—on your own participation; collect all written work; and then reflect alone and with your colleagues about how motion, change, and mathematics might take shape together, in your understanding, in your classroom and your school.

Notes

Workshop 7.

Next Steps

(required for those seeking graduate credit)

You have completed a series of six video workshops in which you observed teachers and students of all ages working on a variety of mathematical problems, and have worked on the same problems yourself. Many of these investigations may have looked different from the mathematics often seen in classrooms. Workshop 7 is intended to help you bring these activities into your mathematics classroom.

Guiding Questions

Philosophy

1. What similarities and/or connections did you notice between the parts of the video involving the pre-K children and those focusing on students in elementary and secondary grades?

2. What similarities and differences for solving the Towers problems did you observe across the various segments: (a) for the children and the adults and (b) for the children over time?

3. What issues may have come up for the teacher/researchers working with the children?

Students

4. How do students use notation as a problem-solving tool?

5. When and how do we see students connecting meaning to the symbols that they use?

6. How might the students' early building be reflected in the work they did in later investigations?

   a. Is it a direct application of ideas built in early grades?

   b. Is it in the way they reason and demand careful explanation?

   c. Is it in the way they rethink, reformulate, and then rebuild what they have done before?

   d. Is it in the way they listen to each other?

7. What might you say about the "flow of ideas" as the students study and work together?

Guiding Questions

Teachers

8. What part does the teacher/researcher play in the building of students' mathematical ideas?

9. How might the task design contribute to:

   a. the building of ideas by learners?

   b. the teacher's role in fostering the building and sharing of ideas?

10. What ideas and issues raised by the video relate to local, regional, or national standards?

11. What implications do you see for your own teaching?

12. Think about and discuss:

   a. something you will change in the way you introduce the next unit you are going to teach.
   b. what you will do to get your students to work together, to share their solutions, and to justify those solutions to one another.
   c. how you will work together with one of your colleagues who has also participated in the workshop.
   d. how you will share your work with other colleagues and your principal.

Notes

Appendix

Index of Reading Assignments

The following required reading assignments for this workshop can be found at the end of this Appendix.

Workshop 1

Davis, R. B., Maher, C. A., and Martino, A. "Using Videotapes To Study the Construction of Mathematical Knowledge of Individual Children Working in Groups." *Journal of Science, Education and Technology,* 1(3), 177-189, 1992.

Workshop 2

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Taping of archival footage from the Rutgers long-term study was supported in part by National Science Foundation grants #MDR-9053597 (directed by R. B. Davis and C. A. Maher) and REC-9814846 (directed by C. A. Maher) to Rutgers, The State University of New Jersey. Any opinion, findings, conclusions, or recommendations expressed in this publication are those of the authors and do not necessarily reflect the views of the National Science Foundation.

Production Notes and Acknowledgments

This series is the result of dedicated work and commitment on the part of many people. Over 450 hours of videotape was gathered, logged, evaluated, and edited to create the finished programs. Much of this material is from the Robert B. Davis Institute for Learning at Rutgers University, which has gathered over 2,000 videotapes of qualitative research footage in mathematics education, dating back to 1988.

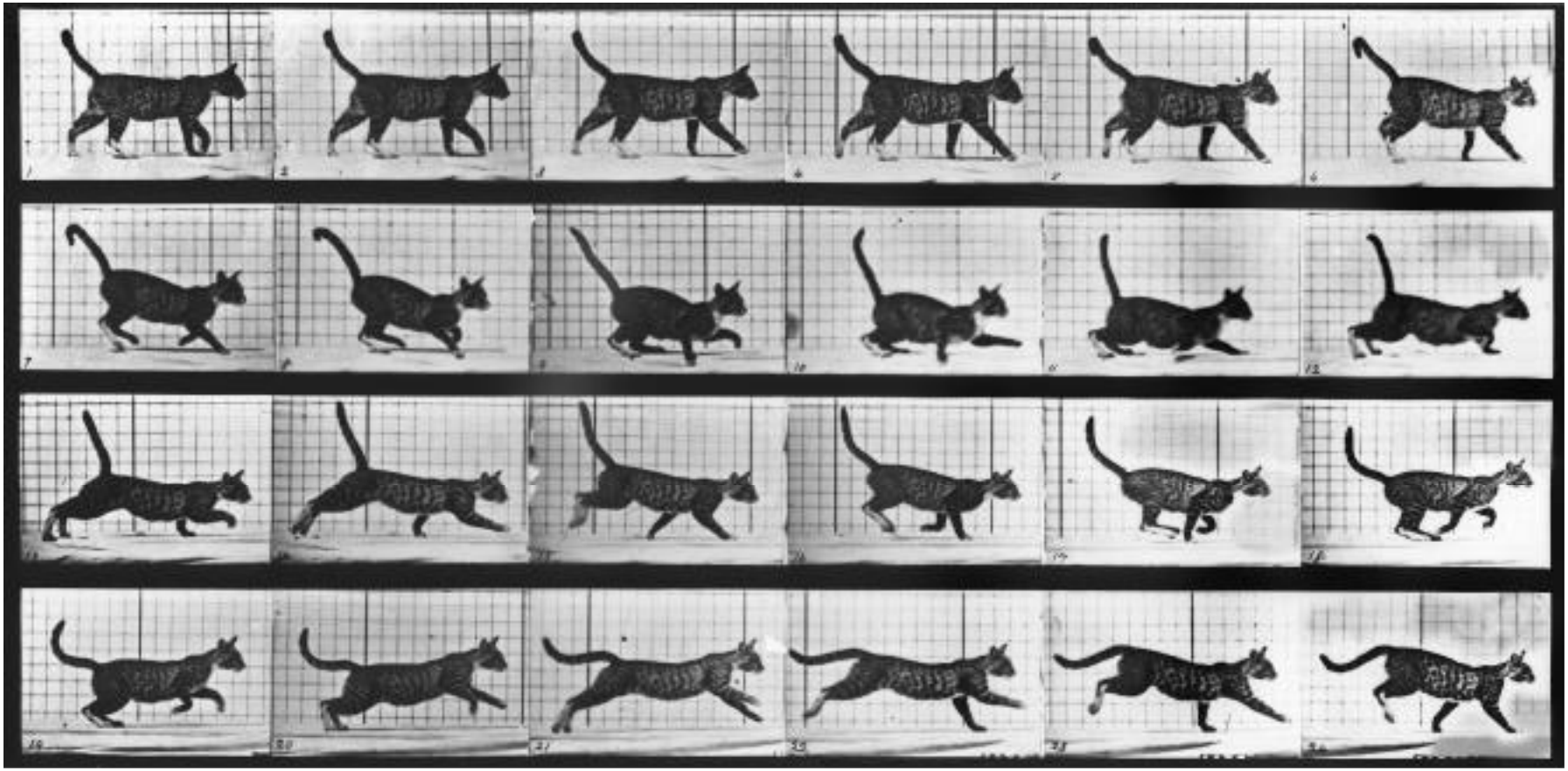
For the most part, this material is preserved in excellent condition, but sometimes the limitations of the equipment available at the time of recording are noticeable as a slight fuzzy quality in the image and sound. In spite of this, we feel that the strength and originality of the student thinking that was so artfully captured overshadow the technical limitations of the medium.

In editing this material into manageable length, every effort was made to retain the integrity, authenticity, and spirit of the students' work. For example, when assembling the archival footage, the editors tried to maintain the original order of shots as much as possible. Furthermore, in the series overall, the scenes in which Kenilworth students appear are placed in the order in which the events depicted actually occurred. For the mathematically minded, the opening and closing music in the series is an original score based on the Fibonacci Series.

In addition to the archival footage, the series includes a great deal of original videotaping in a number of public school districts. The producers wish to thank the Kenilworth, Englewood, Jersey City, and Middlesex public school districts in New Jersey, and the Provo, Utah public school district for allowing access to their schools. The teachers, students, and administrators who participated in the series should be applauded for their willingness to "go public" with their ongoing efforts to raise student achievement in mathematics.

This series is the first comprehensive effort to make available on videotape the fruits of the Rutgers long-term study. Simply put, the career-spanning efforts of Carolyn Maher and her colleagues and collaborators at the Robert B. Davis Institute for Learning are what have made the series possible, and the producers hope that these workshops do some small justice to their long-term efforts.

Finally, all of us who have had the privilege of working with the students in the Kenilworth study appreciate their willingness to take on the challenges of mathematics. It is all too easy to forget that this cohort was randomly selected in first grade. Only the stability of small-town life has allowed so many of this group of students to stay together over such a long period. We only hope that the lessons this wonderful group of young men and women have taught us will be of benefit to other educators who have the vision and courage to help their own students reach for deeper understanding in mathematics.



VOLUME 1, NUMBER 3

SEPTEMBER 1992

JSEEEP 1(3) 149–220 (1992)

ISSN 1059-0145

Journal of Science Education and Technology

PLENUM PRESS • NEW YORK AND LONDON

Journal of Science Education and Technology

Volume 1, Number 3 **September 1992**

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Journal of Science Education and Technology, Vol. 1, No. 3, 1992

Using Videotapes to Study the Construction of Mathematical Knowledge by Individual Children Working in Groups

Robert B. Davis,[1,2] Carolyn A. Maher,[1] and Amy M. Martino[1]

Videotaping small groups of students in a regular classroom environment makes it possible to study individual student cognitive growth in a social setting. The present report deals with student development of some new mathematical ideas over an extended period of time.

KEY WORDS: Mathematics; representation; videotape; elementary.

INTRODUCTION

Videotape may do for mathematics education what the microscope did for biology—it may allow us to see many things that would otherwise remain invisible. By recording (on videotape) the mathematical work of students, following these same students for several years, and analyzing these tapes carefully, it is possible to observe the development of mathematical ideas over time.[3] When—as in the present report—this recording is done in a regular classroom setting, one is able to study individual student cognitive growth in a social setting, and thus gain insight into how social processes influence personal cognitive development.

In the case reported here, a combinatorics task was presented to some second graders. This same task was presented to them again five months later, when they were third graders. We give some data on their behavior in each case and relate their performance to what is known about instruction that took place in the classroom.

The Students

We report on six children, three girls (Dana, Jaime, and Stephanie) and three boys (Brian, Jeff, and Michael). In both second and third grades, mathematics instruction made use of small groups and was based on an expectation of considerable student initiative in devising ways to solve mathematical problems. In the second grade, the students worked in triads; at one table, Dana (D), Stephanie (S), and Michael (M) worked together (we call this group I); at another table, Brian (B) and Jeff (J) worked with Jaime (Ja) (group II). In the third grade, the children worked in pairs. What we will call group III consisted of Dana and Stephanie; group IV, of Jaime and Michael; and group V consisted of Jeffrey and Brian.

The Mathematical Task

The task given to the children was a word problem, for which they were *not* told in advance any method for solution. The problem went as follows:

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[2]Correspondence should be directed to Robert B. Davis, Center for Mathematics, Science and Computer Education, Rutgers University, 192 College Avenue, New Brunswick, New Jersey 08903.
[3]The videotaping and analysis reported here is part of a six-year study of eight children, which is now in its fourth year. The teachers of these children are participants in a long-term teacher development project (Maher *et al.*, 1991). The children attend a K-8 elementary school that serves a working-class neighborhood.

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Stephen has a white shirt, a blue shirt, and a yellow shirt. He has a pair of blue jeans and a pair of white jeans. How many different outfits can he make?

Data Collection

This word problem was given to the entire class, which was organized to encourage individuals and groups to work at their own pace and without teacher intervention. Following the group working sessions, the children were asked to share their group ideas with the entire class. After the sharing session, children were individually interviewed about the problem task. Data for this study came from children's written work and analyses of videotape transcripts from the following three sources: (1) classroom videotaping of the two triad groups (for grade two) and the three pair groups (for grade three); (2) classroom videotaping of each group sharing their solution strategy with the rest of the class; and (3) videotaping of interviews with the individual children following each class session.

THEORETICAL BASIS FOR THIS STUDY

Every study must begin with some theoretical framework, whether explicit or implicit, whether deliberate or unexamined. The conceptual basis for this study is that thinking about a mathematical situation involves cycling through a series of steps (Davis, 1984). First, it involves building a representation of the input data. From this data representation, memory searches are carried out to construct a representation of relevant knowledge that can be used in solving or trying to solve the problem.

A mapping between the data representation and the knowledge representation is constructed, checked out, and if judged to be satisfactory, is used to solve the problem. Moving from a data representation to a knowledge representation that might be useful in solving the problem may not be automatic. As the learner attempts to map the data and knowledge representations, checks are made along the way and other knowledge may be entered. In the process, some representations are rejected and/or modified. The learner cycles through this entire sequence, or parts of it, many times before constructing a "final" method of dealing with the problem. When the constructions appear satisfactory, other techniques associated with the knowledge representation may be applied to carry out the solution to the problem. Examples illustrating how the data representation of the problem statement are gradually constructed by individual students are given in Davis and Maher (1990).

What will emerge as of special interest in the present study are the relatively "primitive" foundation blocks from which these representations can be constructed. The process of building up representations from cognitive building blocks is referred to as "assembly" (Davis, 1984). In general, it is argued that effective building blocks are, at the beginning, the result of *experience*, although this is not always "concrete" experience (but it very often is). Of course, as the use of these blocks develops in more abstract or more generalized situations, more "abstract" building blocks come to be synthesized, first in a metaphoric way, and later in the form of truly abstract characterizations.

One can illustrate this development by considering the case of the concept of "function." In its earliest versions, the idea of "function" is merely a matter of being able to give certain answers and recognizing that these answers are dependent upon something else (as one might be able to give the price for a purchase of pencils, at ten cents apiece, if one knew how many pencils are to be bought). In the course of this work, one begins to use—perhaps even to invent!—some method for writing what mathematicians call *variables.* With more experience, one comes to recognize some of the essential features of tables, graphs, formulas, etc., as representations of functions, and one begins to see what all of these separate instances have in common. "Functions" come to be *things* that one can deal with, and one learns more and more ways of dealing with them. At some later stage, one can reformulate this into something akin to the abstract definition in terms of a set of ordered pairs (and so on).

As this development progresses—and usually *before* one gets to the "ordered pairs" abstract definition—one is likely to draw on a metaphoric use of earlier concrete experience, perhaps thinking in terms of *mappings* or even in terms of pieces of yarn extending from each possible "input" to the corresponding "output." Of course, even the words "input" and "output" themselves represent this use of metaphor, in terms of "putting something in" and "getting something out"—remnants of earlier concrete experience with some kind of gadget or appa-

ratus or arrangement, perhaps like a machine where you toss oranges into a hopper and get orange juice out of a spigot.

Seymour Papert (1980) refers to this process of building abstract ideas on a metaphoric use of previous (possibly concrete) experience when he describes how important for later learning was his early opportunity to play with gears and to observe the way that turning one gear would result in the turning of another, perhaps faster, perhaps more slowly, perhaps in the opposite direction: "I became adept at turning wheels in my head and at making chains of cause and effect . . . . I believe that working with differentials did more for my mathematical development than anything I was taught in elementary school. Gears, serving as models, carried many otherwise abstract ideas into my head." (Papert, 1980, p. vi). We would phrase this differently: Gears, serving as cognitive building blocks, made it possible for Seymour Papert to build mental representations for other ideas that might seem unrelated—and unrelatable—to gears, ideas such as cause and effect, ratio, and so on. This pattern of building upon earlier experience is of particular significance in the present study, because we shall see some children building upon previous experience—using previous experience as *assimilation paradigms* or "internal metaphors"—in situations where most adults expect the children to operate on a far more abstract level. Metaphoric thinking and true abstraction are two very different ways of dealing with mathematics, as a growing collection of instances is making clear. It is important for teachers to be familiar with both types of thinking and to see how one type builds on the other.

GRADE TWO: THE SHIRTS AND PANTS ACTIVITY

The activity in grade two will be reported separately for the two groups of children: Group I consisting of Dana (D), Stephanie (S), and Michael (M) and group II consisting of Jaime (Ja), Jeff (J), and Brian (B). It is important to notice that, although the children did work in groups, their individual ways of representing the problem, and the methods for solution which they invented, were their own, and usually different from those of others in the group. They listened to one another—a little. They argued with one another, but did not usually concede acceptance of another student's point of view. They looked at one another's work, but might—or might not—be influenced by it. These students, it would have to be said, "think for themselves." Perhaps this should not surprise us; after all, what other form of actual thinking is there?

Group I—Dana, Stephanie, and Michael (May 30, 1990)

These children began by focusing on information that dealt with type of clothing and color to build up a representation of the problem situation.

D: I'm just gonna draw a shirt . . . that's all we have to do . . . and then put like. [Dana drew three shirts.]

S: I'm gonna make a shirt . . . and put white . . . wait . . . W for white. [She drew a shirt and placed the letter W inside the outline of the shirt.]

M: Yeah, white shirt, white pants. [He drew a white shirt and white pants.]

They further refined their ideas by considering information about the number of items of clothing, drawing a picture that represented the data.

S: Ok, blue and then a yellow shirt. [Stephanie drew blue and yellow shirts.] He has a pair of blue jeans and a pair of white jeans. [She drew two pairs of jeans.] How many different outfits can he make? Well . . . [Dana looked at Stephanie's paper and drew blue and white jeans.]

M: He can make only two outfits.

The students decide that differences in the kinds of outfits are relevant. Michael's suggestion that there are two possible outfits stimulates Stephanie's curiosity. She cycles back and rereads the problem checking that the input data representation is consistent with her knowledge representation.

S: Well, no [read] how many different outfits . . . he can make a lot of different outfits. Look, he can make white and white . . .

D: He can make all three of these shirts with that outfit. [Dana pointed to her three shirts.]

From this language we might infer that Dana has the key idea for exhausting all possible combinations, but one should not be too quick to reach

this conclusion. We cannot say, this early in our observations, whether Dana is building on her actual experience with "combinations of clothing items" being used to create "different outfits" or whether she is using a more abstract notion of what constitutes an "outfit." Figures 1a, 1b, and 1c show the final written work of Dana (Fig. 1a), Stephanie (Fig. 1b), and Michael (Fig. 1c). Notice that Stephanie uses her diagram to develop a coding strategy to form her combinations:

S: I'm gonna make a shirt [Stephanie began to draw a shirt.] and put white . . . wait . . . W for white. [She then labeled the shirt by writing W inside its traced outline.]

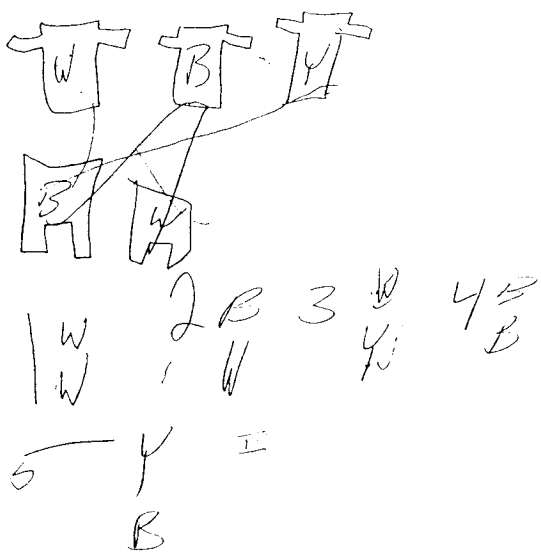

Fig. 1a. Dana's second grade solution.

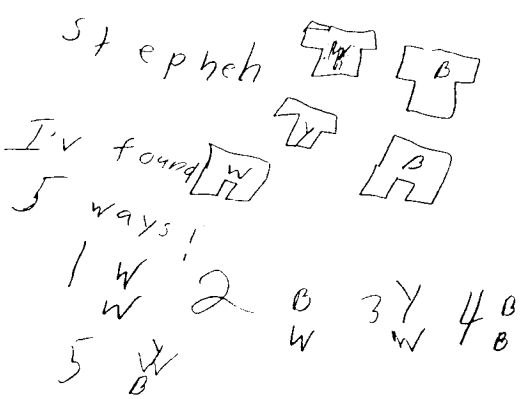

Fig. 1b. Stephanie's second grade solution.

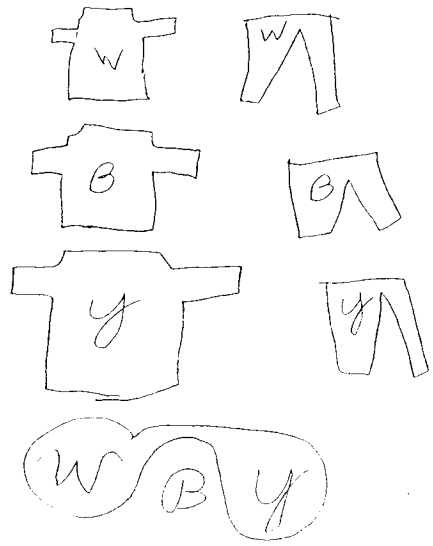

Fig. 1c. Michael's second grade solution.

Stephanie then illustrated each distinct outfit with a pair of letters, the first for the shirt and the second for the jeans. She recorded each outfit and kept track of them by numbering each combination. As she was recording the first outfit, she turned to Dana and Michael and said:

S: You can make it different ways too. You can make white and white, that's one . . . W and W. [She drew a 1 and W over W, meanwhile Michael looked over at Dana's work.]

M: That's what I'm doing. [He erases a piece of his white pants.]

S: Two could be blue...blue jeans and a white shirt . . . blue, W. [She drew a 2 and B over W.]

D: Yeah, well just put white with blue [Dana drew 5 connecting lines between the rows of shirts and pants.]

Dana found a notation that enabled her to make use of her original idea and spontaneously drew lines that connected each of her white and blue shirts to each of her blue and white jeans and her yellow shirt to her blue jeans. She concluded, as indicated in Fig. 1a, that there were a total of five different outfits. Stephanie continued to list pairs of letters for her outfits and concluded, also, that there were a total of five combinations.

S: Ssshhh . . . Ok, yellow shirt . . . number three can be a yellow shirt. [She drew 3 and Y over W.]

D: It can't . . . yellow can't go with the white.

Now, suddenly, we get a revelation! Dana is *not* thinking in terms of some abstract notion of what constitutes a "combination" or an "outfit" ! *She is building on her actual past concrete experience with selecting clothing items that can "go together" in a harmonious way.* (After all, when we say "I love your outfit," we are talking about the way things fit together, and not merely upon the fact that they are present.)

We get further data from Dana's drawing (Fig. 1a), which indicates that she didn't draw a line between the yellow shirt and white jeans. For Dana, an outfit is the kind of combination of clothing items that her experience has taught her to consider appropriate. She appears to ignore Stephanie's remark that the outfit doesn't have to match. Dana has not moved to the stage of thinking about abstract outfits which are to include every possible combination of one shirt with one pair of jeans, however unsightly the result.

During this time, Michael worked quietly alone, occasionally stopping to listen to his classmates, or to talk about his recording of white shirt with white pants and blue shirt with blue pants.

S: No . . . how many outfits can it make . . .?

M: I'm doing white shirt and white pants, blue pants, blue shirt.

S: It doesn't matter if it doesn't match as long as it can make outfits. It doesn't have to go with each other, Dana!

The videotape shows Dana tapping with her pencil and counting her five connecting lines.

D: Four outfits it can be . . . .

S: It can be more if you put them mixed up. Just watch. I'm on my third one right here . . . number four. It could be a blue shirt and a blue pants. [She drew 4 and B over B.] Number five can be . . . a white shirt . . . [She drew 5 and W] . . . Wait . . . [She erased the W.] OK . . . a blue shirt . . . Wait! . . . Did I do blue and white? [Dana looked over Stephanie's shoulder and Michael drew a blue shirt.]

Note that we are able to observe some of Stephanie's self-monitoring or "control" planning, as she checks over whether she "[did] blue and white."

Dana chose not to revise her picture to include the yellow shirt and white jean combination, even after Stephanie commented on the irrelevance of taste. In choosing not to follow Stephanie, Dana is exhibiting typical behavior for these students: They may listen to others, but they are not quick to change their minds. (Of course, if one construes outfit in the terms of everyday experience, as Dana was doing, then taste never is irrelevant.)

Nor had Stephanie herself found all possible combinations, despite the fact that she intended to do so. Her diagram did not include one combination, the white shirt and blue pants outfit. Notice in Fig. 1b that Stephanie's fifth entry shows a letter W between the letters Y and B as she recorded the yellow shirt and blue pants outfit. Observation of the videotape episode indicated that Stephanie first wrote the W, erased it, and then wrote a Y over the B. Dana then recorded Stephanie's solution on her paper beneath her line diagram. At this time, Michael continued to draw without speaking. Continuing:

D. What's two? [Dana was inquiring about Stephanie's second combination in an attempt to compare results.]

S: [Stephanie was not acknowledging Dana's request, and continued with her fifth combination.] It can be a yellow . . .

D: What's two? [She looked over Stephanie's shoulder and began to record her coded combinations. Michael looked up and continued to draw.].

Recall that early in the problem session, Michael reported that he had found two outfits, white shirt with white pants and blue shirt with blue pants. Stephanie tried to convince him that there were more combinations. However, Michael was engaged with drawing his own picture (Fig. 1c), and although he seemed to be aware of Stephanie's coding strategy, he appeared to reject it. Now we see Michael acknowledge Stephanie's strategy, but decide to reject it:

S: Two is blue shirt and white pants . . . a blue shirt and yel . . . wait . . . a yellow shirt . . . did I do yellow and white? A yellow shirt and blue pants. [Stephanie drew Y over B.]

D: A yellow shirt and blue pants.

M: I don't want to do it that way . . . I want to do it this way. [He referred to his own

picture, and explicitly rejected Stephanie's system of coding.]

D: Well, do it the way you want.

S: Do you know what? There's five combinations . . . there's only five combinations. 'Cause look you can do a white shirt with white pants . . . .

Unsurprisingly, Michael's final solution does not resemble Stephanie's. In fact, what is particularly interesting about this classroom episode is that each child produced an independent solution, and each seemed to be satisfied with his or her own solution. Michael's solution is also idiosyncratic in a second way: He has shown three colors of jeans, instead of the two that the problem statement specified. This is a familiar kind of error; we conjecture that a partial representation—in this case, for the three colors of shirts—is inadvertently used in places where it should not have been (in this case, by applying it also to the jeans).

Group II—Jaime, Jeffrey, and Brian (May 30, 1990)

These children began by immediately deciding that they could make two outfits out of the five pieces of clothing.

J: Two 'cause he has a white shirt and a blue shirt and a yellow shirt and a blue jeans and a pair of white jeans. There's jeans and a shirt and jeans and a shirt.

Ja: Two.

Brian expressed concern about the shirts and pants matching.

B: A white shirt and a white pair of pants match, and a blue shirt and a blue pair of jeans match.

So! Brian, too, is building "metaphorically" upon actual previous experience. For Brian, as earlier for Dana, outfits need to match in order to be considered. This is what outfits are in the real world. One might have expected that the children would deal immediately with the abstract idea of "*any* combination of a shirt with a pair of jeans." But if one is aware of the importance of premathematical ideas one is not surprised to find that an outfit is what the child's experience has established it to be—it must match!

Jaime, too, is building upon the real world idea of an outfit:

Ja: Yeah, blue and yellow . . . blue and white, white and yellow go . . . white goes with everything!

Hence, she opened up the possibility of other combinations to which Brian agrees.

B: [To Jeff] Yeah, white goes with everything.

Although the question of what constitutes an outfit may not be settled—the children do not all agree—group II now begins to consider the critical question of how to be sure that you are finding every combination—however inclusive you wish to be:

Ja: Oh, I know how we can do it. [She was trying to find a systematic way of organizing the work.] White shirt with a blue pants . . .

Jaime (Fig. 2b) began to write, and Jeff and Brian stopped to observe what she was doing. Jeffrey (Fig. 2a), then counted the combinations as Jaime listed them verbally.

J: One . . .

Ja: Then a blue shirt with white pants . . .

To form the second outfit, Jaime reversed the two colors.

J: Two . . . that's two . . .

Ja: White shirt with white pants . . .

J: Three . . .

Jaime's strategy of reversing the pattern began to get into trouble when she reached the white and white combination and she could no longer switch top and bottom colors; a worse problem was coming next:

Ja: Yellow with the blue . . .

J: Four . . .

Ja: Blue with the yellow . . .

When Jaime noticed there were no yellow pants, she terminated her reversal pattern. Nonetheless the children continue their search:

J: Five, six, seven! Seven different pairs! First there's white and blue that's one.

B: How we gonna write this down?

Here we see explicit discussion of the question of inventing an appropriate written representation; of course, the matter of inventing notations has been

an important part of this entire episode. Characteristically, each child has invented his own notation, with a modest amount of borrowing ideas from the others.

Notice that the children deal simultaneously with many different matters. This is an almost universal property of classroom discussions of this type. One consequence is that the children's remarks jump from one aspect of the problem to another, then frequently jump back to an earlier matter. Indeed, at this point Brian returns the discussion to the question of whether an outfit has to match. This time, though, Jaime and Jeffrey try to convince Brian that the outfits need not match:

B: A yellow shirt doesn't match.
Ja: Even though it's weird . . .
B: Yellow white . . . yellow blue . . .
J: You have to make it every single . . . anything . . . any combination . . . you have to make all the different combinations with everything.

So we see that Jeffrey is using the abstract idea of every possible combination, whereas Brian still wants the real-world meaning of outfit. Jaime seems to be somewhere in the middle—"even though it's weird," surely a real-world criterion still being employed.

Communication and Sharing

Almost immediately after this discussion, Brian and Jaime are observed watching and listening to the conversation of the Stephanie, Dana, and Michael group. Jaime and then Brian respond.

Ja: They drew it! [She looked at Dana, Mike, and Stephanie.]

Jaime remarked that the other group drew a diagram.

B: They say five combinations!

The children continued to work on their solution:

J: White shirts and white pants . . .
Ja: No, I have a white shirt match a white pair of pants, a blue shirt match a blue pair of pants, a yellow shirt match a white pair of pants . . . .
J: White pants and white, yellow pants and a [unclear] shirt.

Jaime then turned to Stephanie's group and said:

Ja: You still have to write about it!

Stephanie responded with the following interpretation of their task:

S: You don't always have to write about it . . . you can write a picture or a graph.

As the children continued to list their combinations, Jaime and Brian appeared to lose interest in recording all possibilities. An examination of the written work of each child (see Figs. 2a, 2b, and 2c) suggests no evident pattern for generating the combinations, although earlier Jaime had referred to an "opposites" pattern in her discussion with Brian and Jeff. Their strategy appeared to be one of writing whatever came into your head, which in some cases resulted in repeated outfits and the absence of checking for all possibilities. At this point in the exploration, Jeffrey decided to share his combinations with Jaime and Brian.

J: 1, 2, 3, 4, 5, 6. I got 6. I got white and white, blue and blue, yellow and blue, white and yellow, yellow and blue, and white and blue.

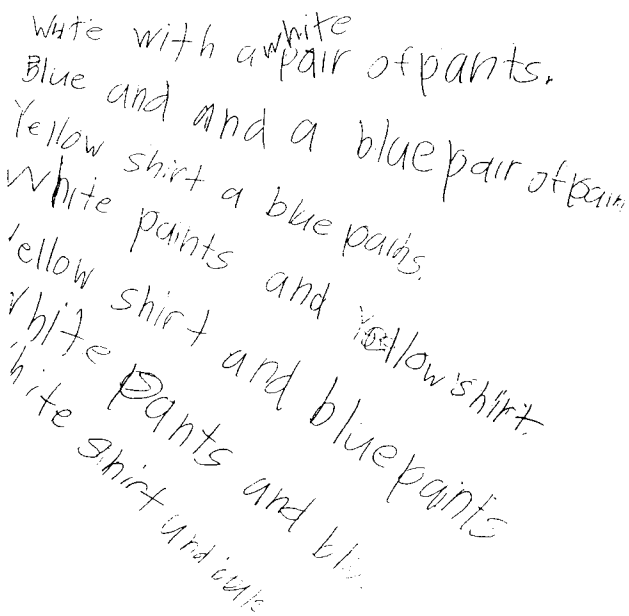

Fig. 2a. Jeffrey's second grade solution.

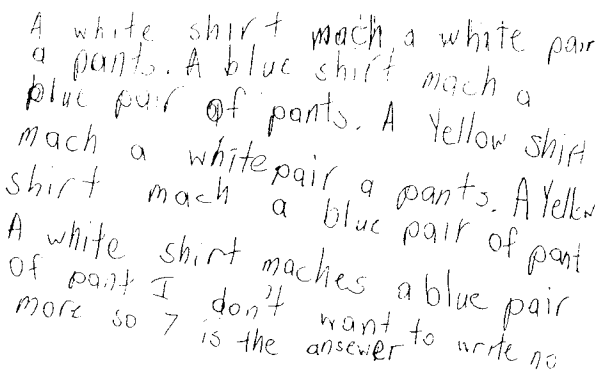
A white shirt mach a white pair a pants. A blue shirt mach a blue pair of pants. A Yellow shirt mach a white pair a pants. A Yellow shirt mach a blue pair of pant A white shirt maches a blue pair of pant I don't want to write no more so 7 is the answer

Fig. 2b. Jaime's second grade solution.

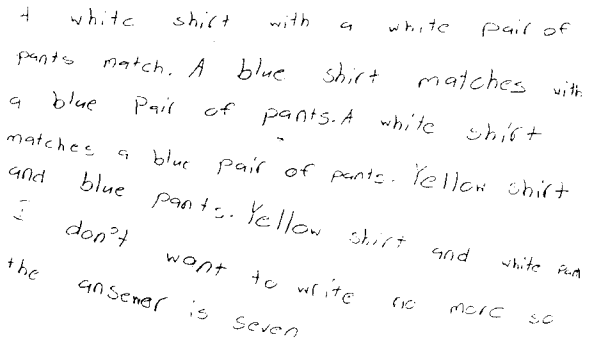
A white shirt with a white pair of pants match. A blue shirt matches with a blue Pair of pants. A white shirt matches a blue pair of pants. Yellow shirt and blue pants. Yellow shirt and white pan I don't want to write no more so the answer is seven

Fig. 2c. Brian's second grade solution.

Notice that Jeff's six outfits were not all different; the yellow shirt and blue pants combination (Fig. 2a) was repeated. Although Jeff read his solution aloud, he nonetheless failed to detect the repetition of the "yellow and blue" combination.

> Ja: I did half of yours already. I put a white shirt matches a white pair of pants, a blue shirt matches a blue pair of pants, a yellow shirt matches a white pair of pants, a yellow shirt matches a blue pair of pants . . .

As Jaime (Fig. 2b) read her solution aloud, we observed that she, like Jeffrey, made use of the doubles pattern of blue and blue and white and white. However, she proceeded systematically to match the yellow shirt with all possible pairs of pants. At this point, the instructor (I) approached the children.

> J: How 'bout white and blue?
> Ja: A white shirt matches a white pair of pants.
> I: How many did you get?
> Ja: I got five!
> B: I got six.
> I: Ok, you gotta find them all.
> Ja: I give up. [She did not write any more combinations.]

Jeffrey showed persistence in completing his list with the replication of the yellow shirt and blue pants combination. His seventh combination was the white shirt and blue pants. Jeffrey's two partners tired of the listing process, and waited for him to complete the list. Brian expressed his loss of interest with this process:

> B: Let's write, I don't want to write no more so the answer's seven.

Effect of Sharing and Communicating within a Small Group

The most evident aspect of the influence of one child's work on another child is that this influence—at least on the surface—is surprisingly small. Michael looks at Stephanie's work and says: "I don't want to do it that way." Figure 2b displays the written work of Jaime, who, like Brian, found five combinations and wrote "I don't want to write no more so 7 is the answer." Note that seven was the answer of another child, so Jaime is in effect saying "OK, have it your own way; I don't really care." This is not agreement, and certainly not an intent to change her own thinking. It is withdrawing from the contest, renouncing the goal of thinking about the matter and trying hard to understand it. Perhaps Jaime, having found five combinations, became uncomfortable when Jeff indicated that there were seven. Jeff's strategy of reversing the order in which pants and shirts appear in his phrases may also have disarmed Jaime and discouraged her from pursuing her own strategy. Another possibility is that having found five combinations, Jaime had an understanding of how to obtain all combinations and simply ceased to record, letting Jeffrey finish the problem.

Whole-Class Sharing of Solutions in Grade Two

Following small group work (from which the preceding transcripts were taken), usual class procedure called for a total class meeting for students to share their solutions. Here, too, individualism prevailed. No singular strategy was adopted for use by the children and no agreement evolved as to the correct number of outfits. Some of the answers re-

ported by different groups were three, five, six, and seven outfits. Generally, the children were pleased to share their representation of the problem solution and talk about how it was derived. Although numerous paths of solution were shared, the apparent result was that no one solution was widely accepted by the students, nor any one "correct" answer.[4]

GRADE THREE

The next opportunity to return to this same problem came five months later, in October, when the children were in grade three. Because both teachers were involved in the study, we know that there had been no class consideration of the problem in the interim.[5] This is of great importance, because one matter this study sought to explore was what happens to ideas in a child's head in the absence of explicit instruction.

Notice what has happened. The children have been given something to think about—first of all, of course, there is the problem itself. Second, there is the influence of the ideas of all of the other children. Even though most of the suggestions from others seemed to be rejected at the moment ("I don't want to do it that way!"), the suggestions themselves may have lingered in the mind, and over the ensuing months they may have taken a firmer shape. If we return to this identical problem now, five months later, will the students show evidence of having somehow incorporated any other children's ideas into their own thinking?

The grade three activity will be reported for the two pairs of students: group III consisting of Dana (D) and Stephanie (S) and group IV consisting of Jaime (Ja) and Michael (M).

Group III—Dana and Stephanie (October 11, 1990)

The following dialogue showed Dana and Stephanie beginning to build a representation for the input data. Dana read the problem aloud. Stephanie responded by suggesting that they draw a picture.

D: Stephen has a pair of blue jeans and a pair of white jeans. How many different outfits can he make?

S: We . . . why don't we draw a picture?

As Dana and Stephanie drew their pictures (Figs. 3a and 3b), we saw them focusing on the pieces of data that dealt with numbers of shirts and pants and their colors. In so doing, they searched for a way to map their knowledge representation to the data representation of the problem:

D: OK . . . He had a white shirt. [The girls drew pictures of shirts.]

S: So, I'll make a white shirt. [Notice that Stephanie was checking her representation as she drew her picture to match the problem data.]

D: A blue shirt . . .

S: I think I'll have to use the big marker for this one . . . you know, color it in blue. [Stephanie made explicit reference to shading the shirt blue to match the problem data.]

D: And a yellow shirt. [The girls drew another shirt.]

At this point, Stephanie (Fig. 3b) suggested that the data be coded by assigning the first letter of the color rather coloring the piece of clothing. Notice that she did not write what she said.

S: Why don't we just draw a Y, a B and a Y [sic] instead of coloring it in?

D: That's what I'm doing.

S: W, B, Y. [Stephanie placed a letter in each diagram of the shirt to represent its color.] OK, he has . . .

D: A blue.

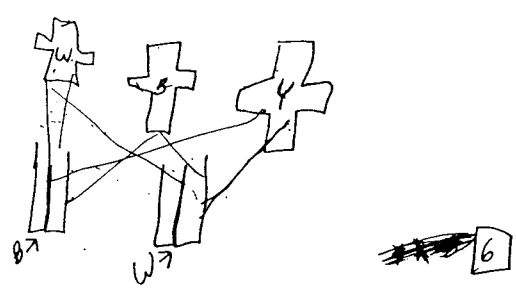

Fig. 3a. Dana's third grade solution.

[4]The lack of closure in reaching a single agreed-upon solution was, in a sense, deliberate. In our classroom work in schools we often find that students may not be ready to agree on a solution to a particular problem task. Rather than reach closure at the time and push for (or even present) a solution, our usual choice is to revisit the same problem at a later time.

[5]Nor had any related material been discussed.

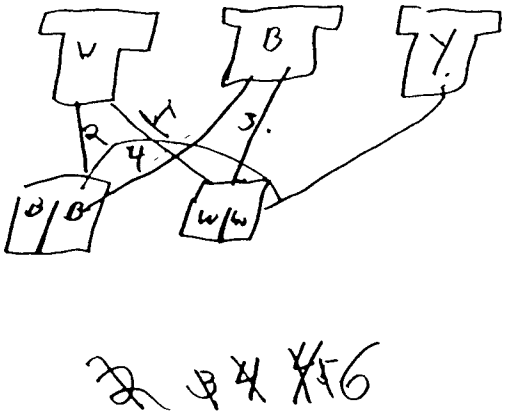

Fig. 3b. Stephanie's third grade solution.

Stephanie reread the problem as she worked to build a representation of the problem.

S: Let me read this . . . . He has blue jeans and a pair of white jeans. OK . . . So let's make blue, blue, blue . . . [She was considering all possible outfits she could make with the blue jeans.]

Dana (Fig. 3a) monitored Stephanie's work, and completed the construction.

D: And a pair of white jeans.

Stephanie indicated that she had begun to construct a representation of the relevant knowledge and proceeded to attack the problem.

S: All right. Let's find out how many different outfits you can make. Well, you can make white and white so that would be one. I'm just going to draw a line . . . . [Stephanie rather than Dana was the person to initiate the connecting line strategy and drew a line from each shirt to the blue pair of jeans then from each shirt to the white jeans.]

Later, Dana and Stephanie were asked by the instructor why they used connecting lines. Stephanie replied:

S: So we could make sure that we were . . . so that we didn't do that again [She referred to repeating a combination.] and say that was 7, 8, 9, 10 [She referred, again, to repeated combinations.] we drew lines . . . so then we could count our lines and say, "Oh, we can't do that again!" or so we could know if we already matched that . . . so we don't go . . . "Oh, OK, that's two [She drew the lines with her finger on the desk top to indicate the two combinations with the white shirt.], that's 4 [She indicated two combinations with the blue shirt.]...and we'd get more than we were supposed to.

Stephanie, in her justification of the use of the line strategy, indicated a shift from working with the representation of the problem data to working instead with a representation of the process by which she solved the problem. Her reflection on that process showed a development to another level of awareness, a shift to a meta level. Whereas at first her pictures represented direct translations of problem data (pictures and colors of shirts and pants), she now invented notation to monitor her own behavior. Notice that her diagram indicated a number label attached to her connecting lines which enabled her to keep track of each combination (Fig. 3b). In this session, neither girl used a coded listing strategy; they simply drew the three shirts and two pairs of pants, connected shirts to pants with lines, and counted their lines. Stephanie explained her preference for the use of lines because they indicated the number of combinations.

We would argue that the meta-concerns that the videotape records are not psychologically different from the direct attempts to draw shirts and pants and to deal with the original problem. The same processes of trying to identify, clarify, and analyze the problem are still in play—it is just that the problem itself has changed. At first, the problem was the one posed by the teacher, about making up outfits of shirts and jeans. Later on new problems have come to occupy the children's attention: how to organize their own work, how to know that they have not omitted any combinations, nor counted the same outfit more than once. The specific tasks are different, but the processes of dealing with them are basically the same.

Group IV—Jaime and Michael (October 11, 1990)

Figures 4a and 4b show the methods of solution used by third-graders Jaime (Fig. 4a), who had previously worked with Jeffrey and Brian, and Michael (Fig. 4b), who previously worked with Dana and Stephanie. The videotape indicated that Jaime

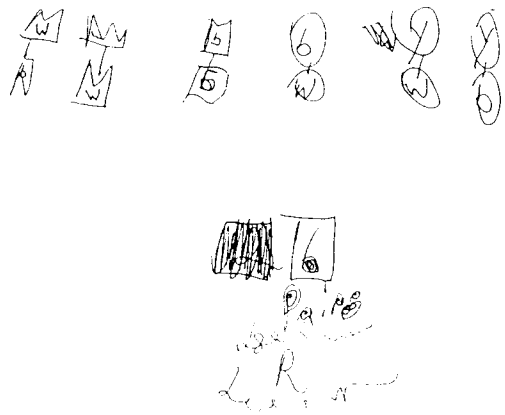

Fig. 4a. Jaime's third grade solution.

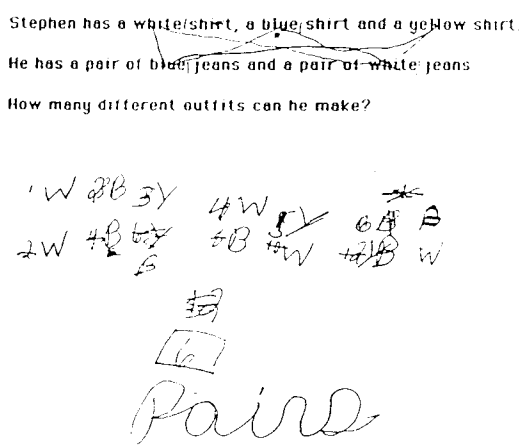

Fig. 4b. Michael's third grade solution.

and Michael worked individually, once again appearing not to listen to each other, each pursuing their own solution. Recall that in second grade: (1) Jaime had recorded her outfits as descriptive phrases, and lost interest after recording five combinations; and (2) Jaime had verbally noted Stephanie's method of drawing a picture to solve this problem.

Now, in grade three, the performances of the children are different. Each child seems indeed to have learned from the others (even though at the time they had seemed not to). Now Jaime draws a diagram (Fig. 4a) that represents each outfit with a two-letter code for color, she shows six shirts and six pants and draws a line to define each outfit as separate from the others. She uses a pattern to generate her diagram which matches each color shirt to both colors of pants before moving on to the next color shirt.

In grade two, Michael's written solution (Fig. 1c) displayed a diagram with three shirts (B,W,Y) and three pants (B,W,Y), with no connecting lines and no numerical answer. In grade three, Michael (Fig. 4b) employs a modified version of the connecting line strategy originally introduced by Dana in grade two (Fig. 1a), and applies it directly to the words in the stated problem. He does this by tracing his finger along the words in the problem to obtain his combinations. However, he records his answers with one-letter color codes, much like those used by Stephanie (Fig. 1b) in grade two. Although it was not obvious that Michael was attending to the representations of his classmates in second grade, his solution strategy in grade three provides strong evidence that he had been somehow aware of their work. Using his new-found strategy, he now achieves all six combinations. Similar results (leading to correct solutions) are shown for all of the children, as a detailed analysis indicates (Maher, Martino, and Davis, 1992a).

WHAT DO THE VIDEOTAPES SHOW?

In general, taped recordings of what students actually do reveal a very great complexity within what was once thought of as the "simple" world of "doing mathematics." Although one can find a seemingly endless collection of themes in these tapes, we will call attention to only four:

Thought in the Absence of Teaching

On May 30, 1990, the children were asked how many different combinations could be made from three shirts and two pairs of jeans. No consensus was reached about the correct answer. The teacher did not insist upon a class consensus, she did not criticize nor evaluate, and she showed neither the correct answer nor any method for solution. Nor did she deal with other similar or related problems. She left the students free to think about the matter if they chose to do so.

On October 11, 1990, the same children, now organized into different small groups, were given the same question. There had been no teaching of this

topic in the meantime. Yet now every student was able to solve the problem, and each felt confident of their answer.

We do not suggest that one should never teach, nor do we suggest that the teachers did nothing. On the contrary, the teachers gave the students "tools to think with." First of all, there was the problem itself. It was intelligible and could be thought about. Second, there was the experience of working on the problem, which brought the students face to face with the need to keep track of their work and to organize it in some systematic fashion in order to make sure nothing was left out and nothing counted twice. There was the input from the other students, which seemed to be rejected, but in fact was not. Especially important, there were the "premathematical ideas" with which the children were already very familiar: shirts, jeans, combinations, counting, different vs. same, and so on. Every child was well able to think in terms of these ideas.

Building Representations in Your Mind

One of the main activities in mathematics is the building up of representations in our minds. What do we do when we listen? We try to build up in our mind a replica (as nearly as possible) of some representation that is in the speaker's mind. This is a very difficult process, as nearly all of our videotapes show only too clearly. In May, when the children seemed not to listen to one another, it would be more accurate to say that they had little success in building up in their own minds replicas of the representations being developed by the other students. They did input something, but it was only five months later that they demonstrated success in building up major parts of these representations. Note that most common theories of teaching pay little heed to the time required to build representations and often operate in ways that make representation building difficult or even impossible (see, for example, Davis, 1987, pp. 110-111).

There is the further question of the premathematical building blocks from which representations are constructed, and the distinction between metaphors based on experience (which probably have an essential role to play, and probably must not be bypassed) vs. subsequent abstract ideas that are constructed after one has used metaphoric assimilation paradigms for an adequate length of time. For some of these children, a true outfit had to represent a harmonious match of colors; only later did they come to the idea of putting things together in every possible way, no matter how unsightly the result. This, after all, is a far more abstract idea (and one that is hardly used in the everyday world).

Questions Left Open and Revisited

During the small-group discussions, we saw time and again that a question would arise, receive some attention, be left unresolved, and be taken up later on (and, once again, be left unresolved). Students seem to have a very large capacity for setting aside unresolved questions, to return to them later, perhaps many times. (Thus, different notational conventions came up and received some thought, with no consensus being reached. The matter would arise again, but—once again—no consensus would be reached. The question of "Must it match?" received this same treatment.) This process of leaving open is probably one more consequence of the difficulty of fitting input data into your own mental representations, which sometimes cannot be done instantaneously.

Capitulation without Agreement

One episode on the tape is perhaps especially interesting. Jaime and Brian, after recording five of the outfits, decided that they did not want to continue. While on the surface it may appear that they were accepting Jeffrey's answer of seven ["I don't want to write no more so the answer is seven."] there are other possible interpretations. Did the students believe the answer was seven? We hardly think so. From their tone of voice and demeanor, it appeared that they were unwilling to pursue what they had come to consider a tedious task. They had had enough! Their written work and conversation indicated that they had begun to understand the problem and were developing a heuristic for carrying it to completion. How is a teacher to deal with this situation? Teachers who believe that learners must build up ideas in their own heads may conclude that insisting upon closure may, in the long run, be ineffective. Others may feel obligated to encourage students to reach closure.

DEPTH OF VIDEOTAPED CONTENT

Videotaping, followed by careful analysis, allows us to see subtleties in student thought processes and to track the development of student thinking over extended periods of time. The depth of content that can be studied in this way is quite remarkable. For this level of detail a price must be paid in terms of time and effort required for analysis. A particularly impressive instance is given in Schoenfeld *et al.* (in press), where a team of researchers spent one and a half years analyzing a three-hour videotape of a single student.

CONCLUSIONS

Mathematics teaching used to be thought of in terms of a teacher standing in front of a class, giving lectures, directions, or explanations to students—or possibly supervising seat work—and methods for studying this process are fairly well developed. In recent years scholars have come to think of mathematics learning quite differently. In the first place, it is now seen as a matter of individual students building up ideas in their own minds (Davis *et al.*, 1990), often constructing abstract ideas by assembling previously learned experiential components (Davis, 1984; Papert, 1980). Social influences are seen as playing an important role (Vygotsky, 1978; Johnson and Johnson, 1991). Classrooms are often organized into small groups of students, working together for cooperative learning (NCTM, 1989). This calls for new emphases on what needs to be studied—what really happens with all of these students working together in small groups? How are ideas built up in the mind of an individual student, when this takes place at least partly in a classroom where other people are involved?

This also calls for a new technology. We argue that videotaping can play a major role in making possible the kinds of observations that are needed in this radically new situation, and in this report we have given several examples.

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ISSN 0104-9720

MEM/USU
— 1998 —

Série Reflexões em Educação Matemática

UNIVERSIDADE SANTA ÚRSULA
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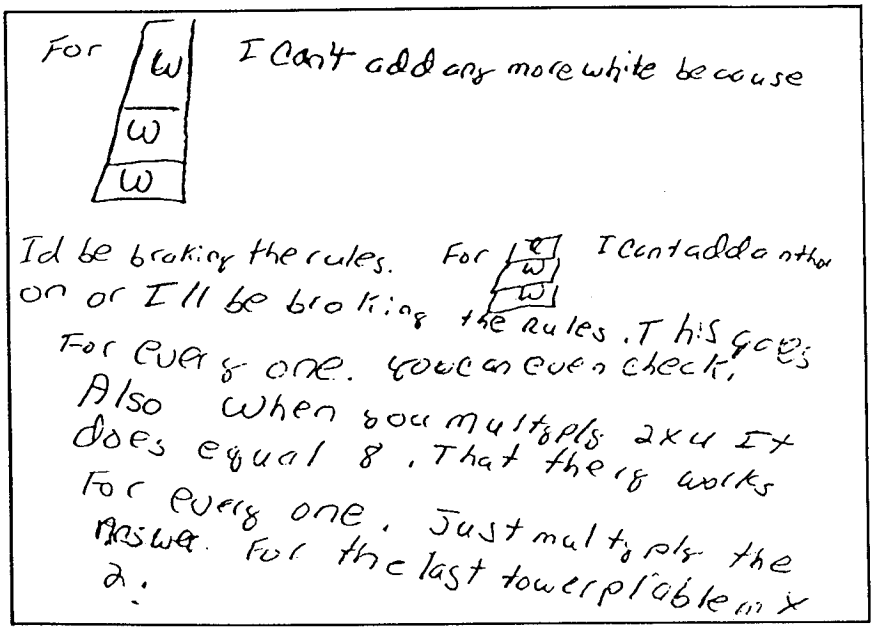
Can teachers help Children make convincing arguments?
A glimpse into the process



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The State University of New Jersey
Graduate Program in Mathematics Education

volume 5

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Chapter Two:

The Nature of Learning

A model for the development of mathematics teachers was proposed and implemented in a K-8 school district in order to bring substantial changes in long established views and practices of teachers (Maher, 1998). The long-term collaboration with the Harding School, in Kenilworth, New Jersey was developed in 1984 and continues today. It was at the Harding School in which the longitudinal study of the development of mathematical ideas in children began with an initial group of children, identified in grade one, and with whom we continue to work as they attend nearby high schools.

The Rutgers-Kenilworth partnership became a clinical investigation of the validity of certain hypotheses of learning, and their application for effective in-service teacher development. The original model for the development of teachers "sought to provide a set of interrelated experiences for teachers to enable them to develop a perspective on mathematics instruction as the creation of a classroom environment in which children construct the concepts and ideas in a problem-solving context" (Maher, 1988, p. 299).

It called for studying, by careful observation and analysis, the mathematical thinking of children who were actively engaged in doing mathematics. The context for that learning required important considerations that were described early in the project:

> In simplest form, it sees learning as arising in a problem-solving context in which students are engaged in investigations in mathematics that give them an opportunity to explore patterns, make conjectures about their character, test hypotheses for their effectiveness in problem solving, and reflect on the formulation of the concept for use in analogous problem-presenting situations. In a constructivist context, important considerations are: (1) motivation as arising from the need

to solve a problem, (2) with the aid of physical embodiments, living through the experience of learning mathematical concepts, operations, patterns and structures, and how they are connected, (3) the extension and development of concepts and operations through generalization in multiple representations of them, (4) the application of the generalization to similar problems, and (5) provision for individual differences in children in the course of problem solving.

(Maher, 1988, p. 297)

Teacher as Learner of Mathematics

The model that guided the long-term intervention in the Kenilworth schools called for teachers to build a broad understanding of the mathematics they were expected to teach, and as their own understanding deepened, "to reflect on the processes by which this knowledge was acquired" (Maher, 1988, p. 300). In a similar way, an emphasis on justification and proof making in schools starts with a parallel emphasis in the proof-making mathematical knowledge of teachers who, like their students, can profit from explorations that evoke model building. Courses and workshops give opportunity for teachers, as learners, to think more deeply about the tasks they are challenging their students to investigate. Teachers, in reflecting on their own solutions and observing the solutions of colleagues, witness and give thought to the variety of representations that are built, as well as to the processes involved in shaping and extending their ideas.

As an example, consider the *Building Towers* problem.[8] Many teacher learners begin, as do young children, by randomly buiding towers. Soon relationships such as pairs of "opposites", that is, opposite colored cubes in corresponding positions, are identified. Checking, in the early building,

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[8] *Brandon's work on towers and pizzas provides another, detailed, example, in Chapter 6. Also, see Davis & Maher, 1993; Maher & Martino, 1997; and Maher & Martino, 1996a.*

is based on recognition of duplicates. Patterns and relationships between towers generate new sets and opposites of those sets. Checking becomes more sophisticated as students begin to monitor for duplicate arrangements by comparing a newly generated tower to ones that have already been built. This leads to other local organizations among sets of towers and ways of generating new towers from existing ones such as "inverting" a tower and its opposite to form another collection. What often follows is a recognition that the new schemes still are inadequate to account for all possible towers. Frequently a search for other schemes leads to a recognition of conflicts between different organizations in which duplicates appeared. Focusing on subsets of these schemes, such as exactly all of a color or one of a color supports the movement to a more global organization.

We have observed that, through investigation and free play, teachers develop organizations that lead to building representations, revising and / or modifying them, sometimes discarding them, and cycling through this process. In so doing, they develop arguments to justify their ideas. During the problem solving explorations, new ideas appear and form the basis and motivation for the development of others. Challenged further by problem extensions, teachers can revisit ideas that, in earlier mathematics learning, were only partially, if at all, understood. By entering new knowledge representations of the input data, deeper understanding can evolve and extensions to other ideas can be constructed. A joy of teaching is to witness the delight of teacher learners who recognize the connectedness of mathematical ideas, such as with the binomial expansion and Pascal's triangle in the *Building Towers* extensions. The notations invented by teachers to express their new ideas pave the way to understanding other symbol systems that were previously meaningless and inaccessible.

Teachers have reported that engagement in these investigations has helped give meaning to the mathematical ideas, connect them to others, and extended them in powerful ways. The elegant and sophisticated constructions produced by students and teachers suggest the appropriateness of these and other tasks for building unification of ideas and concepts, hitherto viewed as compartmentalized and discrete. In fact, very substantial changes have been documented by studies[9] about teaching and children's development of mathematical ideas.

Teacher as Learner of Children's Thinking

After working on the *Building Towers* problem, teachers report that it is helpful to study the varieties of ways that children have thought about the mathematical ideas that were encountered when working on the task. They indicate that a careful study of children's thinking helps them to recognize the variety of diverse representations developed by children. One way to introduce teachers to children's thinking about justification is to view videotapes of children doing mathematics and to study, using transcripts of the tapes and children's accompanying work, the development of children's ideas.

Two tapes that have been successfully used in courses and teacher workshops about proof making in children are *The Gang of Four* and *Brandon and the Pizza Problem*. These are briefly described below. A detailed analysis of the Brandon story is given in Chapter Six.

[9] *See Landis & Maher, 1989 and the following Rutgers dissertations about teaching: Landis, 1990; O'Brien, 1995; Schorr, 1996; and Thatcher, 1995. See also Martino, 1992.*

The Gang of Four, Grade 4

(March 10, 1992)

A videotape shows four nine-year old children, Stephanie, Jeff, Michelle and Milin, seated in a conference-like setting, engaged in a discussion about the *Building Towers* task. In this tape Stephanie tries to convince Jeff that she has found all possible towers 3-tall that could be built, selecting from plastic cubes in two colors, blue and red. In so doing, she swiftly and confidently presents a justification for her solution by organizing the tower combinations into five cases (no blue, one blue, two blue "stuck together", three blue, and two blue separated by one red.) The other children, Michelle and Milin, join Jeff and Stephanie in a very lively and thoughtful discussion. In the same tape, Milin presents an inductive method for organizing the towers.

Notice Stephanie does not use plastic cubes to represent her idea, but rather has invented a notation using letters arranged to represent the various towers. (See Figure 1.)

Individual Written Assessments

(Octuber 25, 1992)

Seven months later, a written version of the *Building Towers Problem* was given to the children. The durability, refinement, and extension of the children's ideas can been seen from their written accounts.

Stephanie. In Figures 2a and 2b, we see a refinement in Stephanie's March 10th representation. She begins by referring to towers 2-tall and writes: "All you have to do is multiply 2 x the number you would get for towers of two. So it is 2 x 4: I will prove it." Notice the care with which she indicates her organization, indicating that she will be classifying according to "color order". She uses the same care to introduce her coding: "R stands for red & W stands for white". The arrangement that is now displayed indicates that the five categories, insisted on in the March 10th discussion, have

been collapsed into four discrete categories (See Figure 2a.). Above each tower representation, she specifies the classification. Stephanie, (See Figure 2b.) then switches to a discussion about the uniqueness of a particular tower. She displays a drawing of a tower with each white cube labeled with the letter W, to indicate a white cube. She proposes that there could only be one 3-tall tower with all white cubes "because, I'd be braking [sic] the rules", that is, she would be contradicting the problem requirement of the given tower height of 3-tall. Stephanie's written work displays not only the durability of her earlier ideas, but also their extension and refinement as new input data, arising out of conversation with her classmates, is entered into her original representation. [10]

Milin. Milin's[11] generalization of a doubling rule is indicated by noting that there would be two towers, 1-tall; 4 towers, 2-tall; 8 towers, 3-tall; and 16 towers, 4-tall (See Figure 3a). He displays drawings of 1-tall and 2-tall towers and labels the blocks using R for red and W for white (See Figure 3b). He produces the two towers predicted by the first doubling, and then the four towers, by the second iteration.

Jeff. Jeff's written work shows a different method for generating all possible 3-tall towers. In his letter to Ronald McDonald, Jeff specifies the notation used to label each tower in the collection. He writes: "W stands for white and M stands for maron [sic]." He, then, displays a drawing of eight towers with each cube labeled using an M or a W. The first tower is all maroon, followed by a tower of all white. Reading from bottom to top, he displays the next three towers with exactly one maroon; reading from top to bottom, the last three towers contain exactly one white. Jeff writes: "I moved maroon up untill [sic] it got to the top then I did the same thing but I used white. The other 2 are all white and all maroon." (See Figure 4).

---

[10] *For further discussion of the development of Stephanie's ideas, see Maher & Martino, 1997, Maher & Martino, 1996a and 1996b.*

[11] *See Alston & Maher, 1993, for a more detailed analysis of Milin's development of proof by induction.*

Brandon and the Pizza Problem

Brandon, a nine-year old boy, worked in his classroom on both the *Building Towers* (November, 1992) and *Pizza Problem* (March, 1993) tasks. A study of the videotape data enables us to observe his construction of an initial representation of a solution to the tower problem and, later, the pizza problem. In a classroom written assessment about the tower problem (December, 1992) and in a follow-up interview (April, 1993), the consistency and stability of Brandon's original constructions are apparent.

Initially in the November, 1992 exploration, Brandon solved the 4-tall tower problem by showing that he found sixteen towers by identifying a tower and its opposite. He said he thought he had accounted for all of them because he could not think of any more. Four months later, Brandon worked on *The Pizza Problem.* He invented a notation of 0's and 1's to keep track of his topping choices and recorded his findings on a table that he built. Brandon's organization enabled him to account, by cases, for all possibilities.

During the April 1993 interview Brandon was asked about his solution to *The Pizza Problem.* When he completed his explanation, he was asked if the pizza problem reminded him of any other problem he had done. Brandon recalled the towers task and immediately sought to find all possible 4-tall towers that coud be built when selecting from two colors. In this second construction, he again found sixteen towers. Again, he organized the collection by eight sets of opposites. He then spontaneously noticed that pairs of opposite towers could be reorganized in such a way as to map into his coding scheme for pizzas.

The videotape interview enables us to observe Brandon explaining to the interviewer his representations to both problems. In so doing, Brandon is able to reflect upon and modify his original problem solving for the tower problem. Brandon, who originally had constructed two <u>different</u>

representations, one for the tower problem and the other for the pizza problem, spontaneously recognized a relationship which enabled him to coordinate them into a single representation. He established an isomorphism between the towers and pizzas, and physically mapped each tower onto the appropriate row of his table of zeros and ones. Brandon, thoughtfully engaged in his problem solving, coordinated the two representations and constructed a more elegant and powerful representation for towers. Brandon's new actions on the towers, driven by his newly constructed representation for pizzas, enabled him to rebuild a representation of the input data for towers. In cycling through this process, Brandon was able to build a convincing argument, according to cases, for both the pizza and tower problems.

Building a personal philosophic perspective

Analyzing the mathematical thinking of children by examining videotape records of problem-solving sessions helps teachers think more deeply about children's learning and the complexity of the development of ideas (Davis, Maher, & Martino 1992). Accompanied by efforts to build a deeper understanding of mathematical ideas, teachers' guided reflection on their own and children's learning will, it is expected, help to develop a personal, philosophic perspective on the learning and teaching of mathematics.

THE GANG OF FOUR

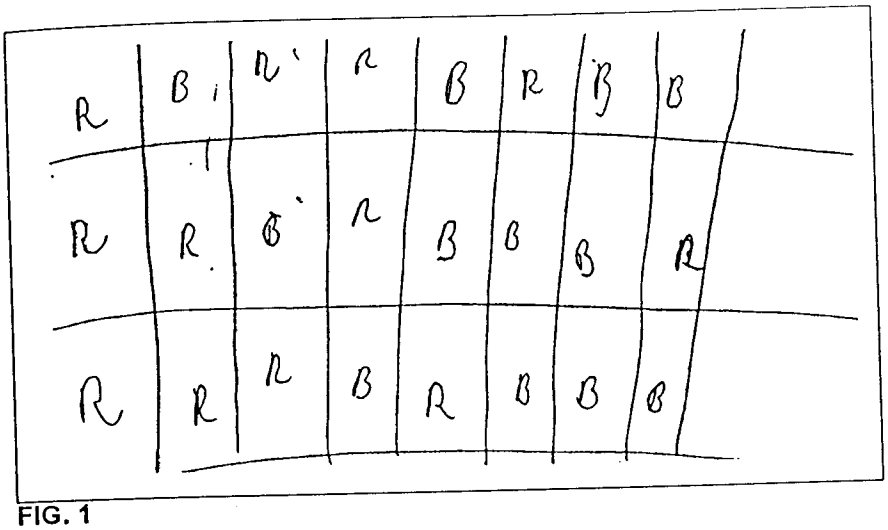


FIG. 1

Name Stephanie Date

Please send a letter to a student who is ill aand unable to come to school. Describe all of the different towers that yyou have built that are three cubes tall, when you had two colors availablle to work with. Why were you sure that you had made every possible tower iand had not left any out?

Dear Laura,

Today we made towers 3 high and with 2 Colors. We have to be sure to make every posible pateren. There are 8 patterns total. I know because all you have to do is multyply 2 X the number you would get for towers of two. so It is 2x 4= I will prove it. IF I put the towers in Colo order The colors are red & whit R stands for Red & W stands for white.

0 Red	1 Red on Top	1 Red in middle	1 Red on bottom	2 Red on top	2 Red on bottom	2 Red on top & bottom	3 Red
W	R	W	W	R	W	R	R
W	W	R	W	R	R	W	R
W	W	W	R	W	R	R	R

IF this doesent Conv, since you I tell you more → over →

FIG. 2a (Stephanie)

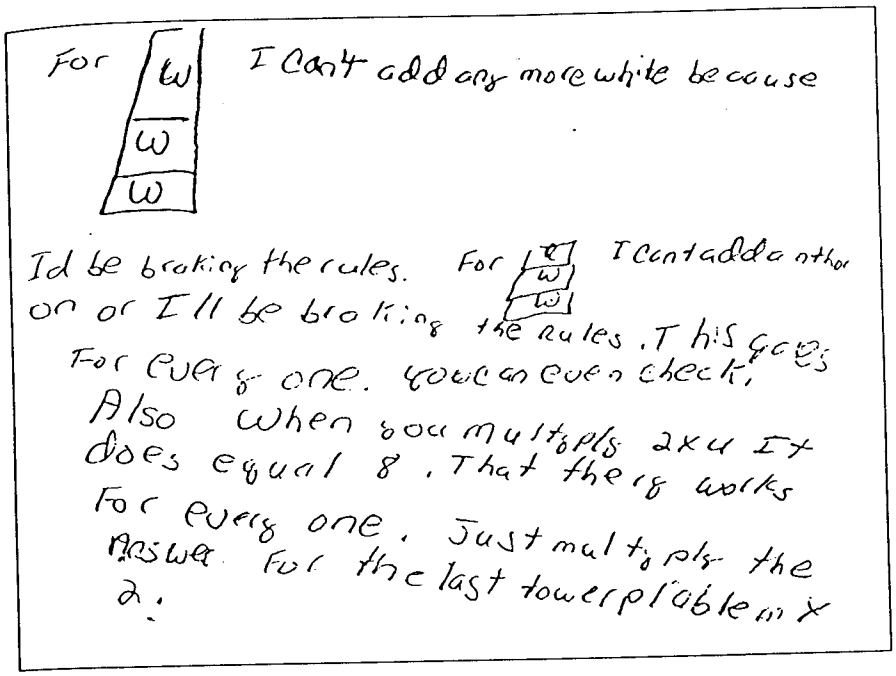


FIG. 2b (Stephanie)

Name Milin Patel Date

Please send a letter to a student who is ill and unable to come to school. Describe all of the different towers that you have built that are three cubes tall, when you had two colors available to work with. Why were you sure that you had made every possible tower and had not left any out?

To a Person,

I drew how many their are because 1=2 2=4 3=8 4=16 5=32 and on the bottom is all of the ways.

R W B

2 3
4
8
16
32

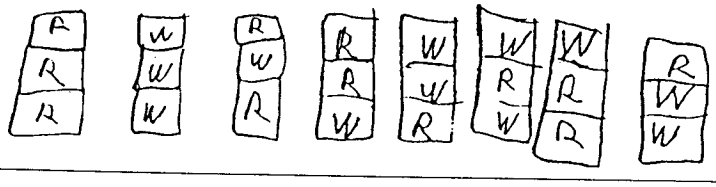


FIG. 3a (Milin Patel)

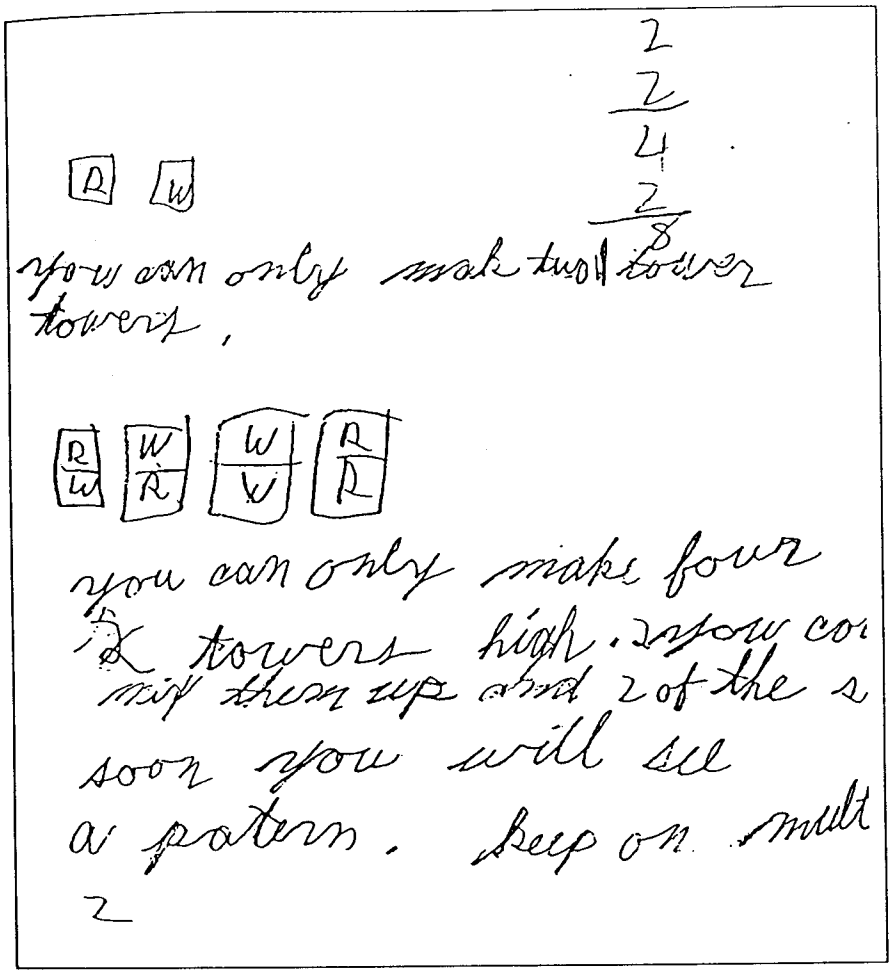
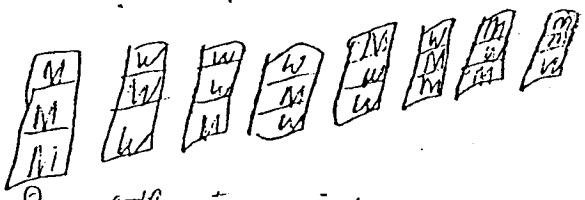


FIG. 3b (Milin Patel)

Name Jeff Goal Date

Please send a letter to a student who is ill and unable to come to school. Describe all of the different towers that you have built that are three cubes tall, when you had two colors available to work with. Why were you sure that you had made every possible tower and had not left any out?

Dear Ronald McDonald,
You missed a great Day of math. Amy came and we worked with unifix cubes. We had to make towers with 3 cubes. Here's how they looked W stands for white and M. stand for Maroon



I used the step method. I moved Maroon up. untill it got to the top then I did the same thing but I used white. The other 2 are all white and all maroon.

FIG. 4 (Jeff)

THEORIES OF MATHEMATICAL LEARNING

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Contents

24 Young Children Invent Methods of Proof: The Gang of Four

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BACKGROUND

This research is part of a longitudinal study of the development of mathematical ideas in children. We are interested in studying how children build mathematical ideas in classroom environments that support children in creating models, inventing notation, justifying their solutions, and participating in mathematical discussions with others. The children in this report have come from such classroom environments since Grade 1.

Our previous findings suggest that after children have built their own representation of a problem task, they seem ready to listen to the ideas of other students (Maher & Martino, 1992a). In so doing, their original ideas may be challenged or supported. The resulting student interactions may lead students to reject original ideas in favor of others, or may enable students to modify, consolidate, or strengthen an original argument (Maher & Martino, 1991). For this reason, after children have developed their own justifications for their problem solutions, we often ask them to try to convince a partner or another small group that their solution is correct. The opportunity for students to test their ideas and hear the ideas of other students provides a setting for the teacher as a moderator and observer to listen to and assess the thinking of the students involved in discussion (Maher & Martino, 1992b). A sharing of ideas between students sometimes takes the form of a small-group discussion that can often be observed by the teacher.

This report describes one such small-group assessment that was videotaped as four children shared their justifications during a combinatorial problem task. A version of proof by cases and a version of proof by mathematical induction were

presented by the children. Important points are highlighted in their discussion and are traced back to earlier work done by the students.

Over the last several years, our work has centered on studying how children build up their ideas when confronted with problematic tasks that promote thoughtfulness about mathematical situations. Tasks are deliberately chosen to challenge children to reorganize or to extend available existing knowledge.

The child, in the process of working out a solution to a problem task, retrieves from memory existing knowledge in the form of mental representations (Davis, 1984). If existing representations are an inadequate match to the problem, the child may find it necessary to reorganize and/or extend his or her available existing knowledge. The process of tackling the problem may trigger the construction of a more adequate representation and provide the individual with incentive to reorganize or extend his or her current knowledge. This process can lead to the development of a new idea (Davis & Maher, 1990).

PRIOR RESEARCH ON METHODS OF PROOF

Much of the research reported on the development of justification and proof has been conducted with older elementary school children, high school students, and adults (Bell, 1976; Galbraith, 1981; Martin & Harel, 1989; Williams, 1980). A considerable portion of this research on older students has focused on the development of formal mathematical proof (Senk, 1985). An exception to this trend is work by Lester (1981) that documented fifth- and seventh-grade students' use of trial and error methods to solve problems and their ability to coordinate multiple bits of information. His work pointed to a marked increase in students' use of a global classification strategy and a dramatic decrease in reliance on trial and error and local classification strategies as children progressed in age. Another study that focused on younger children's development of justification and reorganization of their previous arguments came from Lawler (1980), who indicated that what makes sense to the child dominates what is inculcated as an extrinsic rule. Sometimes it appears that a child has regressed in knowledge; what may at first appear as a decline might instead be interpreted as the result of a struggle between "sense-making" and the input of new information. Lawler's interpretation is the latter. He suggests that when a new "control element" enters the scene, it may subordinate previously independent micro views into a more global and integrated type of knowledge. This process occurs over some time.

Our own observations of children doing mathematics are consistent with Lawler's view. The mathematical strand that we are tracing over time is young children's development of justifications and methods of proof (Maher & Martino, 1991). Balacheff (1988) made distinctions between justifications, proofs and mathematical proofs. He characterized justification as discourse that aims to establish for another individual the validity of a statement, proof as an explana-

tion that is accepted by a community at a given time, and mathematical proof as those proofs accepted by mathematicians. We are consistent with Balacheff's terminology when we refer to justification and proof. The goal of this chapter is to study 9-year-old children's development of methods of justification over time, and to examine how these arguments become refined and accepted by the classroom community to form the basis for a method of proof.

ORGANIZATION OF THIS REPORT

The existence of an extensive library of videotapes and other data on children's performance makes possible a new kind of study, one that enables tracing the development of the details of a child's thinking over several years (Davis, Maher & Martino, 1992). This study comes from such a data collection. Thus, in tracing the development of students' thinking over time, several videotapes are referred to in this report. Analysis of the March 10, 1992, videotape entitled "The Development of Fourth-graders' Ideas About Mathematical Proof," which has also come to be known to many viewers as "The Gang of Four,"[1] provides the major focus. Reference is made to several other tapes (which are outlined in Table 24.1) that were made prior to this March 10 tape in order to trace the early origins and development of these ideas.

TABLE 24.1
Problem Activities and Interviews Used in the Analysis

Grade	*Date Recorded*	*Activity/Objective*
2	May 20, 1990	Shirts and Pants Problem: Students find all possible combinations that can be made when selecting from three shirts and two pairs of jeans.
3	October 11, 1990	Tower Problem 1: Students find all possible towers that are four cubes tall when selecting from plastic cubes in two colors.
3	October 12, 1990	Tower Problem 1, Interview 1: Individual children talk about the combinations they found and present any organizations they may have used to arrange their towers.

(*continued*)

[1]The videotape entitled "The Development of Fourth-Graders Ideas About Mathematical Proof" has come to be identified as "The Gang of Four" since its first presentation by Robert B. Davis at Andrew Gleason's retirement dinner at Harvard University in 1992. This tape is available through Carolyn A. Maher, Rutgers University Center for Mathematics, Science and Computer Education, 10 Seminary Place, New Brunswick, NJ 08903.

TABLE 24.1 (*Continued*)

Grade	*Date Recorded*	*Activity/Objective*
4	February 6, 1992	Tower Problem 2: Students find all possible towers that are five cubes tall when selecting from plastic cubes in two colors.
4	February 7, 1992	Tower Problem 2, Interview 1: In this interview, random guess and check methods are replaced by local organizations as students monitor their production of combinations.[a]
4	February 21, 1992	Tower Problem 2, Interview 2: Stephanie, Milin, and Michelle[b] further discuss their organizations for accounting for all possibilities. Stephanie chooses to explore towers that are six cubes tall.
4	March 6, 1992	Tower Problem 2, Interview 3: Stephanie and Milin further explore a partial "proof by cases."[c]
4	March 10, 1992	"The Gang of Four": A taped discussion of four children that provides the focus of this paper. The four students are asked to find all possible towers of height three when selecting from two colors of plastic cubes. In their attempt to justify having found all possible towers to their classmates, Stephanie shares her version of a "proof by cases" and Milin shares his version of a "proof by mathematical induction."

[a]The progression from use of random methods to the use of systematic "local" organizing aids that serve to simplify the task of finding new towers and keeping track of their collections enables children to find new groupings (e.g., by "inverting" a tower and its opposite to produce a new collection, or by arranging a subset of towers that has exactly one blue cube arranged according to some pattern such as having exactly one blue cube but in a different position). These organizations were discussed and students were further challenged to consider whether they had accounted for all possibilities. One student, Jeff, indicated the pattern that the number of towers doubled each time another layer was added to the towers (two towers one cube tall, four towers two cubes tall, eight towers three cubes tall, etc.). While working with these local organizations, some children came to realize that their local organization schemes were inadequate for accounting for all possible towers. In many cases, children came to recognize conflicts between their different local organizations and realized the need for an overall scheme to account for all possible towers.

[b]Michelle, like Jeff in Interview 1, had discovered the doubling pattern that resulted when the height of towers was increased by one cube.

[c]In the course of Tower Interview 3 conducted March 6, 1992, Stephanie developed a "proof by cases" for towers four cubes tall and recognized the "doubling pattern" to predict the number of towers of a given height. Milin developed a "proof by mathematical induction" to predict the total number of towers of any height by understanding how the doubling pattern worked. For more detail on the development of Stephanie and Milin's proofs over time see Maher and Martino (1993) and Alston and Maher (1993).

"The Gang of Four" (Grade 4, March 10, 1992)

We begin this analysis of Stephanie's problem-solving behavior by considering a videotaped small group interaction in which Stephanie tries to convince one of her classmates, Jeff, that she has found all possible towers three cubes tall that could be built selecting from plastic cubes in two colors, one blue and the other red. Two other children, Michelle and Milin, join Jeff and Stephanie in the discussion. The four children sit in a conference-like setting, observed at a distance by their classroom teacher, the school mathematics supervisor, and research staff from Rutgers. Two cameras and microphones are situated to observe carefully the children and their written work.

When presented with the Tower Problem, Stephanie began by immediately drawing a picture of eight towers. The construction of her first drawing showed the following eight different towers. It took her about 30 sec. to make her drawing (see Fig. 24.1).

The swiftness and confidence with which Stephanie retrieved her set of towers comes with little surprise. Stephanie has been involved in a thoughtful approach to combinatorial counting problems since Grade 2. She and her classmates, students in a school where building meaning in doing mathematics is a serious goal, have benefited from several years of reform in teaching mathematics (Maher, Davis, & Alston, 1991; Martino, 1992).

Classroom Problem Solving Prior to March 10, 1992

In Grade 3, the children worked to find all possible towers that are four cubes tall as a classroom activity (see Table 24.1). They were given Unifix cubes in two colors and were videotaped working in pairs to solve the problem. Following that problem activity, the children were individually interviewed about their solution (October 12, 1990). In Grade 4, the same children were given the problem with the variant of building towers of height 5 (February 6, 1992). All of the children were interviewed afterward (February 7, 1992). Thus, several videotaped classroom episodes and individual interviews of the children provided the data for analyzing how Stephanie built up her idea of proof.

"The Gang of Four" (Grade 4, March 10, 1992)

As referred to in Table 24.1, in this particular session, the goal had advanced from finding all possible combinations of towers that could be built, to providing a convincing argument that every possible tower had been found. The students were asked by the instructor, "How do you know that you have them all?" and "Can you convince me that you have all possibilities, that there are no more or no fewer?"

In response to this challenge, Stephanie made a second drawing of towers that

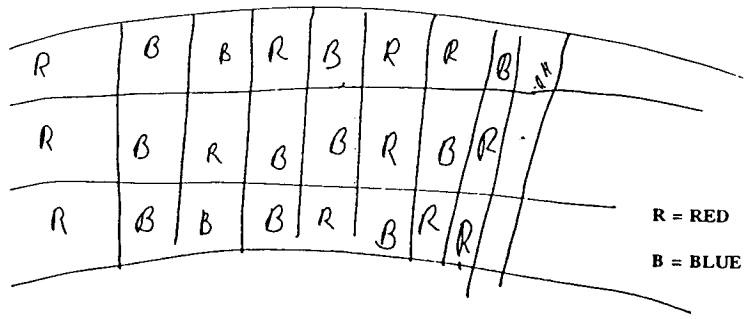


FIG. 24.1. Stephanie's initial drawing of towers.

she used in an attempt to convince her classmates that there were only eight possibilities for towers three cubes tall when selecting from two color choices (see Fig. 24.2).

Stephanie began with the cases of no blue cubes, exactly one blue cube, and exactly two blue cubes that were adjacent.

Stephanie: Alright, first you have without any blues . . . red, red, red [Fig. 24.2, tower 1]. Then you have one blue . . . blue, red, red . . . [Fig. 24.2, tower 2] red, blue, red [Fig. 24.2, tower 3] or red, red, blue [Fig. 24.2, tower 4]. There's no more of these because if you go down another one [She referred to a fourth position in a tower of three.] you have to have another block on the bottom. Then you have exactly two blue . . . yeah, actually that's what I did the last time I was here. You can put blue, blue, red [Fig. 24.2, tower 5] . . . you can put red, blue and blue [Fig. 24.2, tower 6].

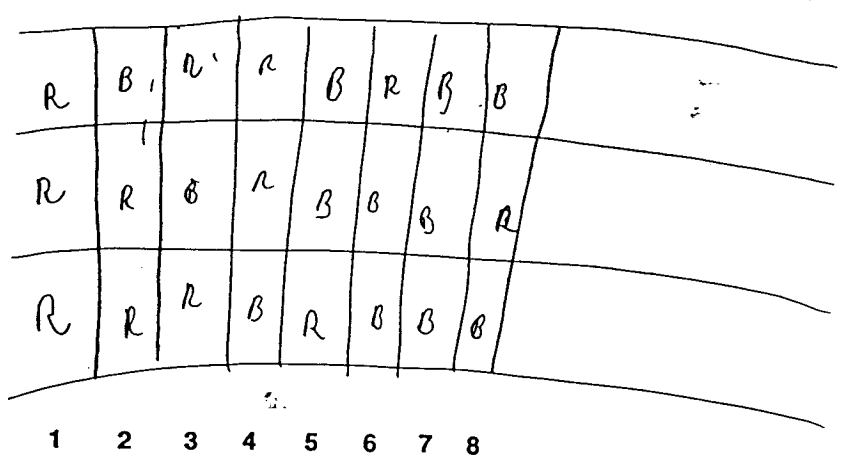


FIG. 24.2. Stephanie's second drawing of towers.

She was then interrupted by Milin, who tried to include the other case of exactly two blue cubes that are separated by a red cube.

Milin: You can put blue, red and blue.

Stephanie reaffirmed her own organization of the data, and was then interrupted by Jeff.

Stephanie: But that's not what I'm doing. I'm doing it so that they're stuck together. [The two blue cubes have no red cubes in between.]

Jeff: You could do one [tower] with one red and then you can do one [tower] with two reds and then one [tower] with three reds and then . . . see, there's all reds, then there's three . . . two reds . . . there should be one [tower] with one red, then you change it to blue.

At this point, the instructor called Milin's organization to Stephanie's attention.

Instructor: Milin said you don't have all two blues, and you said . . . why is that?

Stephanie: [to Milin] Show me another [tower with] two blue [cubes] alright so . . . [She pauses while Milin takes her paper, and then adds the following comment.] with them *stuck together* cause that's what I'm doing.

Milin quickly returned the paper, saying:

Milin: In that case here . . .

Michelle, like Milin, suggested a broader, more all-inclusive organization of cases, that is, all towers with exactly two blue cubes. She added:

Michelle: What if you just had two blues and you weren't stuck together . . . you could . . .

Stephanie again reaffirmed her own organization and responded:

Stephanie: But that's what I'm doing I'm doing the blues *stuck together* [adjacent to each other]. Then we have three blues which you can only make one of . . . then you want two blues stuck apart . . . took apart . . . [separated by a red cube]

Instructor: Separated?

Stephanie: Yeah, separated. Blue, red, blue.

Stephanie had presented a proof for all possible combinations by considering five individual cases (towers with no blue cubes, one blue cube, two blue cubes stuck together, three blue cubes and finally, two blue cubes separated by a red cube.). Another student seated at the table, Jeff, then posed the following question: "Do you have to make a pattern?"

FIG. 24.3. Jeff's use of patterns which resulted in "double counting."

Jeff's Prior Experience with Towers Four and Five Cubes Tall (Grades 3 and 4, October 11, 1990, and February 6, 1992)

On October 11, 1990, and February 6, 1992, both earlier classroom problem-solving sessions building towers, Jeff paid attention to patterns, using them to find sets of towers. In doing so, he constructed different categories in which the same tower appeared. For example, Jeff organized his cubes using the above patterns and consequently unknowingly counted the tower with exactly one blue cube in the fourth position twice (see Fig. 24.3). It is possible that due to these prior experiences, Jeff had become somewhat skeptical about the efficiency of looking for sets of patterns.

Returning to "The Gang of Four" (Grade 4, March 10, 1992)

In response to Jeff's question about making patterns, the other students at the table replied that it was not necessary to make a pattern (in the sense that this was not a requirement imposed by the teacher), but it helped them in various ways. Stephanie responded, "It's easier then just going . . . oooh there's a pattern!" As she talked, Stephanie reached up as if she were grabbing something from the air. Her earlier disequilibrium about the efficiency of using various patterns to solve these problems had led her to consider an alternative all-inclusive organization of the possible outcomes. It was at this time that she introduced a new organization of possibilities and built a proof by cases in which she was now able to consider all possibilities with care to avoid counting particular towers twice.

Looking Back at Stephanie Building Towers Four Cubes Tall (Grade 3, October 11, 1990)

Stephanie and her classmates had worked on building towers four cubes tall in Grade 3 and then five cubes tall in Grade 4. In Grade 3, Stephanie found individual towers by trial and error and guess and check. She built what she thought was a different tower, and then she compared it with others to see whether it was a new one or a duplicate. Stephanie also invented descriptive names like "red in the middle" or "patchwork" to identify individual towers. These names were applied to particular towers that were considered individually and were not used to relate the towers to a group or subclass of towers. Also, in Grade 3, Stephanie tried other strategies in her solution. For example, occasionally a tower and its "opposite" would be positioned next to each other, but no explicit reference was made about this pattern (see Fig. 24.4). (Tower A and

FIG. 24.4. A pair of towers that Stephanie named "opposites."

Tower B are opposites if, for all n, the cube in position n in tower B has the opposite color from the cube in position n in tower A.) Although Stephanie used a trial-and-error method for generating towers and referred to "opposites" in Grade 3, she did not implement widespread use of this "opposite" pattern in the construction of towers until Grade 4.

Also, in Grade 3, Stephanie noticed another relationship between two towers formed by flipping the first tower to make the second, naming the second tower "cousin" (see Fig. 24.5). (For towers with height h, tower B was the cousin of tower A if for all n, the cube in position n in Tower B had the same color as the cube in position h-n in tower A.)

Stephanie: Let me check. Nope. Nope . . . these must be cousins. [She referred to two towers, one tower with exactly one blue cube in the bottom position and one tower with exactly one blue cube in the top position.]

Instructor: You think they're cousins . . . why do you think they're cousins?

Stephanie: Because this one has one [blue cube] on the bottom and this one has one [blue cube] on the top . . . turn it [the first tower] upside down and they're the same [the two "cousin towers"].

Looking Back at Stephanie Building Towers Five Cubes Tall (Grade 4, February 6, 1992, and February 7, 1992)

In Grade 4, this procedure of constructing a tower and its "opposite" became a primary strategy for both Stephanie and her partner, Dana, as they generated original combinations. They then incorporated the Grade 3 relationship of "cousins" (a tower and its upside-down partner) into their strategy for generating new towers. Stephanie explained how she and her partner would build a tower (call it A), build the opposite of A, build the "cousin" of A, and build the "opposite of the cousin of A." Figure 24.6 depicts the four towers that Stephanie and Dana used to demonstrate their "upside down and opposite procedure" to the instructor for towers five cubes tall.

In grade 4, we see Stephanie applying the "upside down and opposites" pattern to group towers into sets. The patterns initially discovered in Grade 3 now formed a basis for building other more complex configurations in Grade 4.

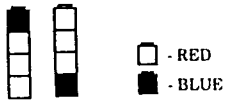


FIG. 24.5. A pair of towers that Stephanie named "cousins."

FIG. 24.6. Stephanie and Dana's use of an "upside down and opposite" procedure.

The use of "pattern" for Stephanie and her partner, Dana, had two different meanings in Grade 3. One referred to arrangements of cubes within a particular tower ("patchwork," "mixed colors," etc.); the other referred to relationships between two or more towers. To find a new arrangement within a particular tower, Stephanie began by using a guess-and-check strategy. She later found sets of towers that were in some way related. For example, she displayed one pattern among towers five cubes tall that had exactly one blue cube placed in each of the five possible tower positions. Another display illustrated the following arrangement: exactly 1 blue cube in the first position, exactly 2 blue cubes in the first and second positions, exactly 3 blue cubes in the first, second, and third positions, exactly 4 blue cubes in the first four positions, and exactly 5 blue cubes. She soon realized that a tower with exactly one blue cube in the first position was counted twice. This produced some uncertainty and suggested to Stephanie that the production of patterns was not always a reliable way with which to find unique towers. This disequilibrium caused Stephanie to consider an alternative argument.

Returning to "The Gang of Four" (Grade 4, March 10, 1992)

During the group's discussion about the usefulness of patterns, Michelle also responded to Jeff's question about patterns. She explained how patterns provided a system of organization for generating new towers and helped to indicate ways to exhaust all possible combinations. However, she was referring to Stephanie's new organization of considering cases.

Michelle: Because if you just keep on guessing like that . . . you're not sure if there's going to be more.

Stephanie then explained to Jeff that Michelle and Milin had a different system for organizing their combinations:

Stephanie: Like Shelly and Milin's pattern was to put this [a tower with blue cubes in the top and bottom tower positions and a red cube in the middle tower position] in a different category.

In an exasperated tone of voice, Jeff responded, "I *know* their pattern."

Looking Back at Milin and Michelle Building Towers Five Cubes Tall (Grade 4, February 21, 1992, and March 6, 1992)

There was evidence from the earlier interviews that Jeff's conviction as well as Michelle and Milin's argument had merit. For example, after the Grade 3 and Grade 4 problem activities, in a series of subsequent interviews, Michelle and Milin had independently developed a method for computing the number of possible towers with a choice of two colors of cubes when given the number of blocks that could be used in each tower. (If n = the number of cubes in each tower, 2^n is the total number of towers.) Thus, from each tower of two, two towers of three can be made, one with a red cube on top and one with a blue cube on top. Previously, both Michelle and Milin had used patterns like "opposites" as strategies to generate the new towers, but each had independently discovered the shortcoming that this method does not help you decide when you have found every arrangement.

Returning to "The Gang of Four" (Grade 4, March 10, 1992)

Michelle, a problem-solving partner with Jeff building towers five cubes tall, also struggled in dealing with the occurrence of duplicates. She now joined in support of Stephanie as she answered Jeff's question about the advantage of finding patterns.

Stephanie: What I'm saying is it's easier to work with a pattern then to say . . .
Milin: [joined in] Here's another one . . .
Stephanie: [She acted out that she was finding individual towers by reaching out in space to imaginary towers.] Yeah that looks good.
Michelle: Cause you might have a duplicate [arrangement in your set] and you may not know.

Another difficulty experienced by these students as they worked in earlier sessions on variations of this activity (building towers of four, five, and in the case of Stephanie, six) was the formation of duplicate combinations. Until they developed systems for organizing their combinations, they frequently had undetected duplicate combinations.

Stephanie: It's harder to check just having them come from out of the blue.
Jeff: [He further challenged their arguments.] How do you know there's different things in a pattern? [The reader is encouraged to refer back to Fig. 24.3.]
Milin: See. Look at this. [Fig. 24.7]

In an impatient tone of voice, Jeff responded: "I *see* that." Milin continued.

FIG. 24.7. Milin's method for finding all towers (a).

Milin: These are all different, right? [He was referring to four towers which he drew that were two blocks high. He then pointed to his drawing of a tower two cubes tall with the red cube on top and the blue cube on the bottom.] From this you can make two more, there's a blue, red, and then blue . . . [indicating all possibilities for a tower two cubes tall when you add a red or blue cube to the top of each tower]

Michelle validated his argument and pointed out that the building up strategy by multiples of two was indeed exhaustive.

Michelle: Cause there's only two colors [of cubes to place in each tower position] more so you know you can't make more.

Milin then drew as he spoke (see Fig. 24.8).

Milin: And then there's blue, red, red [tower with a blue cube in the bottom position] and you can't make any more from this one so [he referred to the tower with red on top and blue on the bottom] you go on to the next one [referred to another tower of height two].

Jeff then questioned Milin: "How do you know you *can't* make anymore?"

Looking Back at Jeff's "Building Up" Strategy (Grade 4, February 7, 1992)

Jeff's question suggested his attempt to reconcile two different systems, the first to find the total number of towers and the second to account for all possibilities. In an earlier interview, Jeff had independently noted the pattern of 2^n combinations to which Michelle and Milin referred. He gave no evidence of being able to imagine what all these towers would look like. For Jeff, this occurred in a subsequent problem-solving session 2 months after the March 10 session when he was introduced to a system of recording using a tree diagram when building upward. At this time, Jeff said:

Jeff: You multiply by two, the last number you got you multiply by two because you make branches off of them. [He referred to adding two branches to the top of each tower to represent the two possible colors which could be added to the top a tower.]

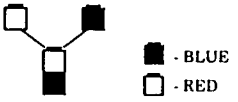


FIG. 24.8. Milin's method for finding all towers (b).

Returning to "The Gang of Four" (Grade 4, March 10, 1992)

Stephanie, in a second attempt to convince Jeff that all possibilities could be accounted for, discarded her drawing with her first set of towers (see Fig. 24.1) and produced eight new arrangements for her towers, appearing rather confident that all possibilities had been found (see Fig. 24.2). Stephanie had drawn a diagram of towers that she had organized in an attempt to convince Jeff that she had found all possible combinations. She did this case by case. She presented five possible cases for arrangements of towers: towers with no blue cubes, towers with exactly one blue cube, towers with exactly two blue cubes "stuck together," towers with exactly three blue cubes and towers with exactly two blue cubes separated. Note that although adults were present in the classroom and listened to the ideas presented by the students, there was no adult intervention in this episode.

Stephanie: Start here . . . okay . . . you have the three together. [She referred to a tower with three red cubes.] You have the [towers with] one blue . . . how could I build another [tower with] one blue?

Stephanie referred to her drawing of four towers (see Fig. 24.2): one with no blue cubes, one with a blue cube in the top position, one with a blue cube in the middle position and one with a blue cube in the bottom position (Fig. 24.9).

Jeff responded: "You can't." His tone indicated that he had considered the cases with one blue cube and emphatically agreed with Stephanie's conclusion that there were exactly three of these towers. In a tone that demonstrated confidence, Stephanie continued:

Stephanie: Okay, so I've convinced you that there's no more [towers with] one blue [cube].

Jeff: Yeah. [Jeff's affirmation of Stephanie's statement was emphatic.]

Stephanie provided further justification for the case of towers with exactly one blue cube by pointing to the three towers with exactly one blue cube, saying:

Stephanie: Blue, red, red . . . red, blue, red . . . red, red, blue, but then how am I supposed to make another one once the blue [cube] got down to my last block [the bottom position]?

Jeff answered in a tone that indicated that he was convinced, "You can't." Stephanie argued that only three towers could be built with exactly one blue

FIG. 24.9. A model of Stephanie's proof by cases for no blue cubes and one blue cube.

FIG. 24.10. A model of Stephanie's proof by cases for towers with no blue cubes, exactly one blue cube, and exactly two blue cubes adjacent to each other.

cube, challenging Jeff to create another tower once all three positions had been utilized. Jeff agreed with her reasoning and conceded that all the towers three cubes tall with one blue cube had been considered.

Stephanie: Okay, so I convinced you that there's no more [towers with] one blue.
Jeff: Yeah.
Stephanie: Okay, now [towers with] two blue. Here's one, right? We have blue, blue, red . . . red, blue, blue, but how am I supposed to make another one?

Jeff then suggested another tower with exactly two blue cubes, saying "Blue, red, blue." Stephanie responded with an explanation that she was considering only those towers with the two blue cubes adjacent to each other, saying, "No this is *together*. Milin gave me that same argument." (See Fig. 24.10.)

Earlier, Stephanie had presented two blue cubes separated by a red cube as a separate case. Milin had objected to her choice of grouping and suggested that all the towers with two blue cubes be considered together. During this discussion, it became apparent that Jeff's use of patterns was different from Stephanie's as he suggested that she group towers as follows: "There should be one [tower] with all reds, then one [tower] with two reds then there's one with one red and then you change to blue."

It is interesting that the patterns suggested in this session were the same as those that Jeff made during the Grade 4 problem activity of building towers five cubes tall, which resulted in duplicates from intersecting sets. Michelle, who was Jeff's earlier partner, also had to deal earlier with the occurrence of duplicates from intersecting sets. Now she assisted in interpreting Stephanie's method for grouping towers.

Michelle: She means stuck together.
Jeff: It doesn't matter.
Stephanie: Stuck together.
Jeff: I know.
Stephanie: Okay, so can I make any more of that kind?
Jeff: No.

Jeff quickly looked at Stephanie's drawing (see Fig. 24.2) and agreed that she could make no more towers with two adjacent blue cubes. Michelle then referred to Stephanie's next case of one tower with three blue cubes.

Michelle: Then you have to move to [a tower with] three [blue] which you can only make one. [Fig. 24.2, tower 7]

Stephanie: Yeah, you can only make one. And then you can make without red which is three blue.

Michelle: And then you can make two split apart.

Stephanie continued and included one tower with the two blue cubes separated by a red cube. [Fig. 24.2, tower 8]

Stephanie: Two split apart which you can only make one of and then you can find the opposites right in the same place [She slapped her hand down on the table.]

Jeff: Okay. [His tone indicated that he understood and acknowledged her method.]

Stephanie: So I've convinced you that there are only eight?

Jeff: Yes. [emphatically]

Looking Back at Stephanie Building Towers Six Cubes Tall (Grade 4, February 21, 1992)

Components of the development of Stephanie's argument can be traced over 3 years and across several problem-solving episodes to include classroom small group work and individual task-based interviews. Limitations in space do not permit a more detailed analysis of approximately 8 hours of videotape data. However, some particular aspects are of interest. For example, in Grade 4, Stephanie generated her towers with the pattern of "opposites." In an interview after the problem activity, she initiated work with towers six cubes tall and brought in pictures of towers with height six that she had drawn at home. Her strategy was to group these by a characteristic such as "all towers with exactly one blue cube" and she then immediately built the "opposite" set of this group. This method frequently resulted in the duplication of arrangements. As she built the towers six cubes tall with two red cubes, Stephanie noticed that these were the same intersection of sets as the towers with four blue cubes. Thus, she had discovered that the "opposites" were also within this method of solution by cases. (see Fig. 24.2).

CONCLUSIONS

For Stephanie, we notice that her ideas began to develop when her earlier strategies no longer served her well and the need arose to invent new ones. We saw that her argument using cases was evident in earlier problem-solving activities. Stephanie first had to build up her ideas in her own mind before she was able to retrieve them so confidently to demonstrate her thinking to her fellow classmates. We interpret this as evidence that Stephanie is building up an assimilation paradigm (Davis, 1984) for proof by cases.

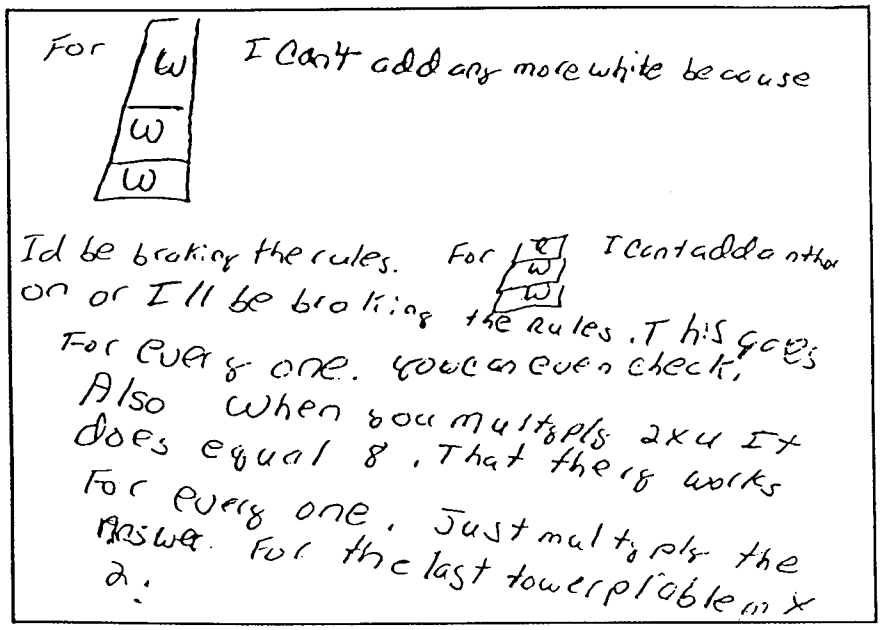
ISSN 0104-9720

MEM/USU
— 1998 —

Série Reflexões em Educação Matemática

UNIVERSIDADE SANTA ÚRSULA
MESTRADO EM EDUCAÇÃO MATEMÁTICA

Can teachers help Children make convincing arguments?
A glimpse into the process



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Chapter Six:

Brandon's proof and isomorphism

Carolyn Maher & Amy Martino

Brandon and Justin Building Towers

(November 17, 1992)

When Brandon and his partner, Justin, were first challenged to solve the *Tower Problem* for towers 4-cubes tall, they generated new towers by trial and error, then built a "partner" tower for each new tower. Sometimes they made an "opposite" partner and other times an "upside-down" partner. (See Figure 1).

The boys, seated as partners, nevertheless worked separately building different "partner" groupings. Eventually, they recognized that using two different "partner".groupings sometimes resulted in producing duplicate towers. (See Figure 1). They adjusted for this difficulty by discarding the "upside-down" pairing strategy and grouped solely by "opposite" pairs. This procedure resulted in eight pairs of "opposite" towers. (See Figure 2).

Brandon Writes About the Tower Problem

(December 17, 1992)

In a classroom written assessment one month later, the children were asked to find all possible towers 3-tall when selecting from two colors of plastic cubes, and to individually justify in writing that they had accounted for all possibilities. Brandon produced a written justification where he again used the organizing strategy of pairs of "opposites" as a form of argument to show all possibilities. (See Figure 3)

Brandon and Colin Working on the Pizza Problem

(March 11, 1993)

Four months after the administration of the *Tower Problem*, Brandon worked with a new partner, Colin, to find all possible pizza combinations

when selecting from four toppings. He used the heuristic of constructing a chart and developed a notation for representing pizzas when he was asked to make pizzas selectiong from four toppings. He assigned the digit "zero" to represent the absence of a topping and the digit "one" to represent its presence. This notation, like other children's use of checklists and numerical codes ultimately aided Brandon in developing a method of proof (Maher, Martino, & Alston, 1993).

Once Brandon developed a notation for recording his combinations, he began to generate pizzas by guess and check methods and recorded each different combination in his chart. Within this chart the beginning formation of local organizations was apparent. For example, three pizzas with exactly three toppings were reported together. (See Figure 4.)

Teacher: What are you doing here, Brandon?

Brandon: Making a graph just like Colin. I put peppers, sausage, mushrooms and pepperoni down and have them like 1, 0, 1, 0 and put... and make a graph.

A portion of Brandon's initial graph is pictured in Figure 4.

Teacher: What does that mean 1, 0, 1, 0?

Brandon: Well instead of using [writing] like pepper down or sausage... you just gotta put like a "1" for yes it's going in, and a "0" for no it's not.

Teacher: So then this first pizza has what on it?

Brandon: Peppers and mushrooms... and then you could have all [toppings] or zero [toppings]... just a plain.

Teacher: What would you write for an all plain one [a pizza with no toppings]?

Brandon: 0, 0, 0, 0.

Teacher: Are you doing something like that too, Colin?

Colin: Yeah.

Brandon: Yeah, he's doing it similar.

Colin had also created a chart, but was using check marks to represent the inclusion of a topping and leaving blank the column(s) in which a topping was excluded. (See Figure 5)

Brandon decided to make a new chart and reorganized his solution using a series of groupings: two toppings, all toppings, three toppings and no toppings. Brandon's second organization is displayed in Figure 6. When he was challenged by his teacher to develop an argument which would show that every possible pizza had been considered, Brandon reorganized his pizzas into an arrangement which took the form of a proof by cases. Brandon's third attempt at building a chart began with the symbols (1, 1, 0, 0), which he quickly changed to (0, 0, 0, 0), as indicated in Figure 7.

Brandon: Colin, you can just have one of each. Colin... you could do this... [He records the four possible pizzas with exactly one topping which appear as entries 2 through 5 on Figure 7].

Colin: I have 15 so far... I think I have... I have 15.

Brandon: Fifiteen?

Colin: Yeah, see look there's the half, the pepper alone, the the pepper and sausage, then the pepper and mushroom, then the pepper and pepperoni, and I put sausage...

Brandon: Oh Colin... Colin... I just noticed... ya know how you're going like that [points to the columns from peppers towards the right] then you could start from the middle and work your way down like that [points to the columns from mushroom towards the right]. That's how I did something like this...

Thus, Brandon generated all the two-topping pizzas by combining

the toppings in the first column [peppers] with each of the three remaining toppings [mushroom, sausage and pepperoni], then combining the topping in the second column [mushrooms] with each of the remaining two toppings [sausage and pepperoni] and the topping in the third column [sausage] with the remaining one topping [pepperoni]. Note how Brandon attempted to match his notation to the notation of his partner.

Colin: Oh yeah, the middle...

Brandon: Wait... I'm gonna try... 0, 0, 1, 1, ya got 0, 0, 1, 1? [Brandon's notation] Look... blank, blank 1, 1. ["Blank" is Colin's method of indicating the absence of a topping].

Colin: Uh Hmm. I have every one.

Brandon: That's it.

Colin: Fifteen

Brandon: Okay... fifteen?

Brandon counted the eleven combinations he had represented in his new chart. These included: the pizzas with no toppings; then, exactly one topping; and then, exactly two toppings. He then predicted that there would be more than fifteen possibilities since the number of choices for zero, one, and two toppings [1 choice, 4 choice and 6 choices, respectively] continued to increase.

Brandon: Eleven. And I didn't even get to three [He refers to including the pizzas with three toppings.]... Colin, I didn't even do three's and I'm into 11 [combinations] so there must be more than 15 [possibilities]. Okay, watch... [Brandon continues to record combinations for exactly three and four toppings arriving at a total of 16 possibilities.]

Towards the conclusion of the activity, Brandon and Colin compared

and checked their sixteen solutions. Colin, in trying to make sense of Brandon's representation, attempted to translate Brandon's code.

Colin: Peppers, sausage and mushroom?

Brandon: Peppers, sausage and mushroom?

Colin: You have two of the same [duplicate pizzas].

Brandon: Oh no, no I checked the wrong one...

Colin: Yeah, that's pepperoni and that's pepperoni and mushroom... [Colin points to Brandon's paper.] and sausage and mushroom and pepperoni... wait, do you have a double? [Colin searches Brandon's chart.]

Notice that in building their arguments the boys refined their notational systems. The framework that evolved was then used as a structure to justify that all combinations of toppings had been considered.

In the process of searching for all possible pizza choices when selecting from four pizzas toppings, Brandon had invented a code to represent the presence of specific toppings on a pizza. He constructed a chart with four columns, one column for each of the four toppings, and represented the toppings as follows: "P" for pepper, "M" for mushroom, "S" for sausage and "peponi" for pepperoni. To represent a pizza, Brandon decided to place either a 1 or a 0 in each of the topping columns to indicate whether or not a topping was present on the pizza under consideration.

Thus, when he placed a "0" in the peppers column it meant that the topping peppers was not included on the pizza under consideration, whereas a "1" in the sausage column meant that the topping sausage was included on the pizza. Using this system of coding, Brandon developed a *proof by cases* for all possible pizzas selecting from four toppings. His five cases included the following: (1) zero toppings [a plain pizza with cheese and tomato sauce], (2) one topping, (3) two toppings, (4) three toppings, and (5) four toppings.

An Interview with Brandon

(April 5, 1993)

When interviewed about his method of proof three weeks after working on the Pizza Problem, Brandon retrieved his former notation of zeros and ones and began to organize his possibilities. Initially, he organized his work by starting with the pizza with no toppings, then making a pizza with toppings A, followed by a pizza with toppings A and B, followed by a pizza with toppings A, B and C. After Brandon had listed fourteen pizzas on his chart, he decided to draw another graph in which all pizzas with n toppings (n = 0, 1, 2, 3, or 4) were grouped together. He was then able to prove that he had accounted for all possible cases, and was further able to demonstrate how he had accounted for all possible pizzas within each case. (See Figures 8a & 8b)

> **Brandon**: Nothing on a pizza... [places a "0" in each of the four topping columns of his chart to represent a pizza with no toppings] this you can have one pepper on a pizza with nothing else... [places a "1" in the pepper column of his chart and zeros in the other three columns to represent a pizza with peppers only] one mushroom on a pizza with nothing else... [places a "1" in the mushroom column and zeros in the other three columns] and you could have a couple of sausages on the pizza with nothing else... [places a "1" in the sausage column and zeros in the other three columns] maybe a couple of pepperonis... [places a "1" in the pepperoni column and zeros in the other three columns].

Brandon continued to organize his pizzas by cases and recorded each with a four digit sequence of zeros and ones. Here he considered

the group of six pizzas which he called "two's". (See entries 6 through 11 in Figures 8a & 8b).

Brandon: And if you don't want any of that [pizzas with no toppings or one topping] you can start getting fancy, and start going to "two's", a pepperoni [sic, pepper] and mushroom and nothing else, then pepperoni [sic, pepper] and sausage and nothing else, then pepper and pepperoni and nothing else... you could have mushroom and sausage and nothing else.

Int: How did you know to go to mushroom now? [See entry 9 in Figure 8a.]

Brandon: Why didn't I use pepper anymore?

Int: Yeah.

Brandon: Cause I already used pepper there [with mushroom]... it's mushroom and pepper [refers to entry 6 on Figure 8a]. And if I put a "1" down for mushroom and pepper that would be the same thing... So each time you go three, two, one...

Brandon explained that he could combine the first pizza topping [peppers, in this case] with the three remaining toppings, the second topping [mushrooms] with the two remaining toppings and the third topping [sausage] with the one remaining topping [pepperoni].

Brandon: You get three choices with the first one [the topping peppers], then with the second one... with mushrooms you only get two choices cause there's only sausage and pepperoni, then with sausage you can only do pepperoni.

As the interviewer probed further, Brandon reaffirmed his strategy

for finding all possible pizzas with two toppings. In the process of explaining his method, Brandon articulated an understanding of distinction between a selection situation in which the order of the items in each subset would not affect the total number of choices [combinations] and a situation in which order would affect the total number of choices [permutations]. He recognized that the Pizza Problem was a situation dealing with combinations and illustrated his point by explaining that having an airplane and a car was the same as having a car and airplane, regardless of the order of the two items.

Int: What I don't understand is... when you move to [the] mushrooms [column] why can you put it with sausage and pepperoni, but you can't put it with peppers?

Brandon: Cause that would be a same thing... if I do that and I put a "1" there [in the peppers column] I've already got pepperoni [sic, peppers] sausage... that would be a same thing. It's just like saying you have an airplane and a car... saying you have a car and an aiplane... it's still the same thing.

Making a Connection

Using his system of coding and organization by the number of toppings, Brandon was able to provide an inclusive organization by cases for his toppings and account for all possible pizzas. When the interviewer asked Brandon if the problem reminded him of any others that he had worked on in the past, Brandon recalled the *Towers Problem* given four months earlier.

Int: In any way does it remind you of any of the problems we've done?

Brandon: It kind of a little reminds me of the blocks [towers] cause

my partner and I... whoever it was... I think it was Colin... did them in order like... yellow, red, yellow, red and then switch around and do the "opposite"... Red, yellow, red, yellow. It's kind of like what you do here [He refers to the case groupings for pizzas.] Just do it in groups... that's what we did with the blocks... how many ways can you make towers... how many ways can you make pizzas... the same problem, sort of.

Int: The same kind of thing? Do you remember how many towers there were?

Brandon: I think I can remember...

Brandon took red and yellow cubes and quickly reproduced eight towers of height four as four pairs of "opposites". Recall that an organization by "opposites" was his strategy for the *Tower Problem* four months earlier when he worked with his partner Justin and three months earlier on his written assessment. It is interesting that Brandon remembered his prior organization for towers, but did not recall with whom he had worked to develop that organization.

When asked to defend how the organization by "opposites" was an effective method to account for all possible towers, Brandon paused for about one minute. He appeared to be studying the chart he had invented to organize his pizza combinations. Brandon attempted to make another chart, this time for towers. He began with two columns and labeled the red and yellow to represent the colors of the cubes. However, he quickly discarded this representation and returned to the plastic cubes and began to build new arrangements.

Brandon built one pair of "opposite" towers with three cubes of one color and one cube of the opposite color, and then built the remaining towers with three yellow cubes and one red cube. He did this by moving

the one red cube up one position at a time starting with the tower with the red cube in the bottom position. He then built the "opposite" set of towers with one yellow cube and three red cubes by moving the yellow cube up one tower position at a time. (See Figure 9).

Brandon completed his task of organizing the eight towers in Figure 9 into two staircases, each with four towers. He then verbalized his recognition of a connection between the tower and pizza problems, and enthusiastically shared with the interviewer his "new" discovery.

Brandon: It's kind of like the pizza problem... like this would be the one's group. [referring to the eight towers of height four with three cubes of one color and one cube of the opposite color]

Int: Let's see.

Brandon: Oh, yeah... I see! This now... this is like the one's group [referring to the eight towers in Figure 9].
You only have one of the opposite color in these [towers with exactly one red or one yellow cube]...
This isn't how I did it, but I just noticed this...

Int: This is very interesting.

Brandon: I just noticed it... then you would have... that would be the one's group, you only have one in there.
Then... in pizzas [holds up the all red and all yellow towers] this would be like the "whole" group or "all" group... save that for last. Now you have the "two's" group... twos [He referred to the six towers of height four with two red cubes and two yellow cubes]. You have two of the opposite color in there.

For Brandon's new organization of towers, he moved from 8 sets of paired "opposites" to 3 sets of towers: (1) the "all/whole group" which

included the two towers with cubes all the same color, (2) the "one's group" which contained the eight towers with one cube of one color and three cubes of the "opposite" color, and (3) the "two's group" comprised of six towers of height four with two cubes of each color. He eagerly talked with the interviewer as he worked. His new organization placed "opposite" towers within the same set. (See Figure 10).

Int: Okay, so you said these [six towers of height four with two red and two yellow cubes] all have two... two yellows or two reds?

Brandon: They had two of each color... they must to be in the "two's" group... if they had three of a color and one of the opposite color they'd be in the "one's" [group]... and you won't have any "three's" group [for towers]...

Int: Why not?

Brandon: Cause for the "three's" group you have three yellows and one red... that would be the same [pointing to the eight towers in his "one's" group].

A "three" group is like that group [pointing to the "one's" group] cause three of the opposite color... So a "three" group would be the same as a "one" group.

Int: Now if I wanted to call all this [the eight towers in the "one's" group] a three's group...

Brandon: Yeah, you could call it a "one" or a "three" group.

Int: Why?

Brandon: You could call it a "three" group cause it has three [cubes] of one color and one [cube of the] opposite [color], or you could call it a "one's" group because it has one of the opposite color. You can call that "three" or "one" it doesn't matter which.

To determine whether Brandon would recognize the structural isomorphism between finding all pizzas selecting from four toppings and finding all towers four cubes tall selecting from two colors, the interviewer asked Brandon to focus his attention on one color within the towers and then to reconsider his groupings.

Brandon decided to place his attention on the yellow cubes within each tower, and changed his groupings of towers from three cases to five cases: (1) towers with no yellow cubes, (2) towers with one yellow cube, (3) towers with two yellow cubes, (4) towers with three yellow cubes and (5) towers with four yellow cubes. (See Figure 11.)

It was at this point in the interview that Brandon, enthusiastically, expressed that the group of four towers with exactly one yellow cube were like the four pizzas with one topping in his chart, and placed each tower on top of its corresponding pizza on the chart. He explained how the red cubes in each tower corresponded to the "zero's" on his pizza chart and how the yellow cubes in each tower corresponded to the "one's" on his chart. He then confidently proceeded to match each of the sixteen towers to each of the sixteen pizzas represented on his chart.

Int: Now, what if we put our focus on say the yellow [cubes]... if we're looking at these eight towers here and we're looking at yellow, and we're looking for the "one's", what would be a ones tower?

Brandon: A "one's" yellow tower?

Int: Yeah.

Brandon picked up a tower with exactly one yellow cube in the second from the top position and responded "That would be a "ones" yellow tower and a three red tower." The interviewer asked what else would be a "one yellow" tower. Brandon's response was to build a staircase pattern with the four towers with exactly one yellow cube. As he arranged the staircase,

he said: **"It's like the pizza problem... you work your way down.** Like pepperoni", pointing to the tower with exactly one yellow cube in the top position. He continued: "mushroom", pointing to the second from the top position, and "pepper", indicating the bottom position. He then picked up his pizza chart and said: "You start with zero."

When the interviewer asked, "What would the "zero one" [pizza with no toppings] look like if we're looking at yellow [cubes]?", Brandon immediately responded by knocking down all sixteen towers and pushed them to the side of the table. The interviewer replied, "I don't understand". Brandon smiled and said, "Blank". Suggesting that such a tower / pizza correspondence could not be represented.

As Brandon realigned the four towers with exactly one yellow cube, the interviewer asked. "Tell me again how this is like the pizzas?". Brandon explained, "You have the one pepperoni... since we're looking at yellow [cubes], the yellow cube would be 1 and the red would be 0". He compared the red cubes in the towers to the digit "0". He continued, "You could have one pepper... then it's like stairs if I draw a line down here like this... it's sort of like here you'd have one pepperoni, one mushroom, one sausage and one pepper." As he spoke, Brandon pointed to the one yellow cube in each of the four towers. Thus, he had noted the similarity between the four towers of height four with exactly one yellow cube and entries two through five in his pizza chart, and then traced a diagonal line on the chart to show the similarity. (See Figure 8a.)

Int: Is what you're saying to me that a yellow cube is like a number "1" in you chart?

Brandon: Yeah... if you're focusing on red, then red would be number "1". Now you would work with the "two yellow group"...

He then took each tower with exactly two yellow cubes and placed it

above the corresponding zero / one code on the pizza chart establishing a one-to-one correspondence between each of the six "two-topping" pizzas and the six towers of height four with exactly two yellow cubes.

Brandon: Right here... you would have the yellow stand for one. So it would be yellow for one, red zero, yellow one, red zero.

Int: I see.

Brandon: That'd be another one.

Int: So this would come next.

Brandon: ... yellow, red, red, yellow. Yellow, red, red, yellow.

Int: I see. So what would this pizza look like? This one? [referring to a tower with yellow cubes in the top and bottom tower positions]

Brandon: That would be pepperoni and... pepper and pepperoni.

Brandon then compared the four possible pizzas with three toppings to the four towers with exactly three yellow cubes. Brandon took the tower with exactly four yellow cubes and called it the "pizza with everything" or four toppings, and concluded that "since the red [cubes] would stand for zero this [the tower with four red cubes] would be the "zero guy", or the tower with no yellow cubes.

Int: You know what I'm wondering? We have this guy left, right? [the tower with four red cubes]

Brandon: Yeah, cause we're not focusing on red.

Int: If we had to call him a name though...

Brandon: Oh, oh... this would be the zero. Oh yeah. Since the red would stand for zero, this would be the zero guy.

Int: This is neat. This is really neat, Brandon.

Brandon: Finally found out what the [all] red [tower] would be... the zero guy.

Brandon also articulated the symmetry of his argument and explained that if he had chosen to focus on the red cubes within each tower instead of the yellow cubes, the yellow cubes in each tower would correspond to the "zeros" on his pizza chart and the red cubes in each tower would correspond to the "ones". He then reorganized his *proof by cases* for towers to show towers with (1) no red cubes, (2) one red cube, (3) two red cubes, (4) three red cubes and (5) four red cubes.

Int: Could we have done it the other way around? Could we have focused on red and gotten it to work the same way?

Brandon: Same way. Now the red [cubes] would be one and the yellow [cubes] would be zero.

Brandon's insight, in recognizing the structural relationships between the tower and pizza problems, came about by his active building and rebuilding of representations, over an extended period of time, in situations that encouraged communication and thoughtfulness.

For each problem task, Brandon and his classmates worked for approximately one and one half hours, over two consecutive days. The children's explorations included two problems, towers and pizzas. The charge was to find all possibilities and to <u>convince</u> us that all were found. The children were not expected to be immediately successful. A four-month time period was allotted for the investigations, so that earlier ideas could be revisited, connections could be discovered, and extensions be made. The teacher did not tell the children whether their solutions were correct or incorrect. The intent was clearly to give this responsibility to the children whose task was to decide, based on the arguments they put forth.

Whether some, or indeed all of these conditions, contributed to Brandon's success is not certain. What is clear, however, is that Brandon was doing mathematics under these conditions and that he was doing it very well.

BRANDON

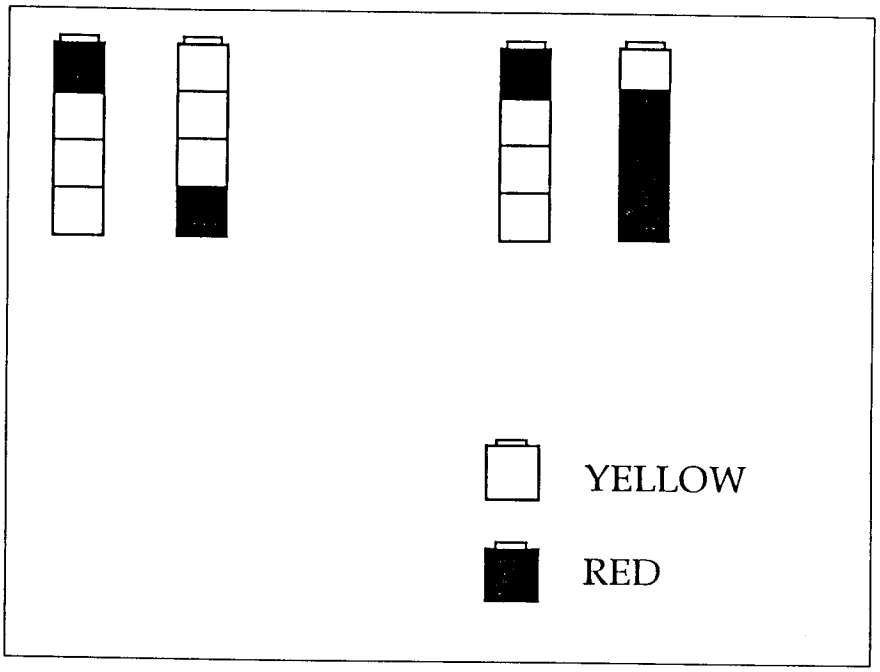


FIG. 1

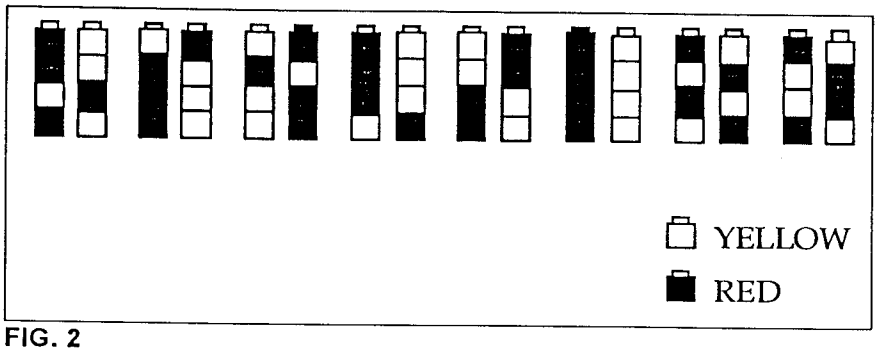
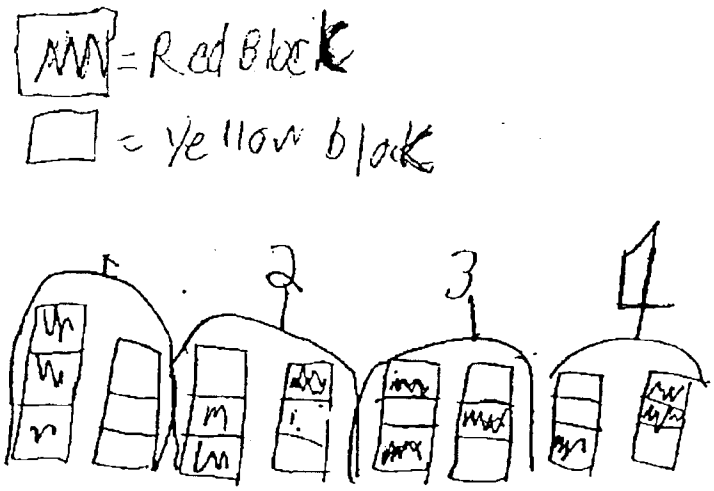


FIG. 2

Please write a letter to someone who has never done this activity. Describe all of the different towers that can be built that are three cubes tall, when you have two colors available to work with. Explain in your letter why you are sure that you have made every possible tower and that there are no duplicates.



Dear ?

I'm in Mrs Zallys' math class and we are bilding. tower
There are 4 different oppisit
pares.

FIG. 3

P	S	M	Peponi	[sic]
1	0	1	0	
1	1	1	1	
1	1	0	1	
1	1	1	0	
0	1	1	1	
1	0	0	1	

FIG. 4

	P	S	M	Peponi
1st.	✓	✓		
2nd. pizza	✓			
3rd. pizza	✓		✓	
4th. pizza	✓			✓
5.		✓	✓	
6.		✓		✓
7.		✓		
8.			✓	
9.			✓	✓
10.				✓
11.	✓	✓	✓	✓
12.	✓	✓	✓	
13.		✓	✓	✓
14.	✓		✓	✓
15.				

FIG. 5

P	S	M	Peponi [sic]
1	0	0	1
1	1	0	0
1	0	1	0
1	1	1	1
1	0	1	1
1	1	1	0
1	1	0	1
0	0	0	0

FIG. 6

P	M	S	P
1.0	1	0	0
2.1	0	0	0
3.0	1	0	0
4.0	0	1	0
5.0	0	0	1
6.1	1	0	0
7.1	0	1	0
8.1	0	0	1

FIG. 7

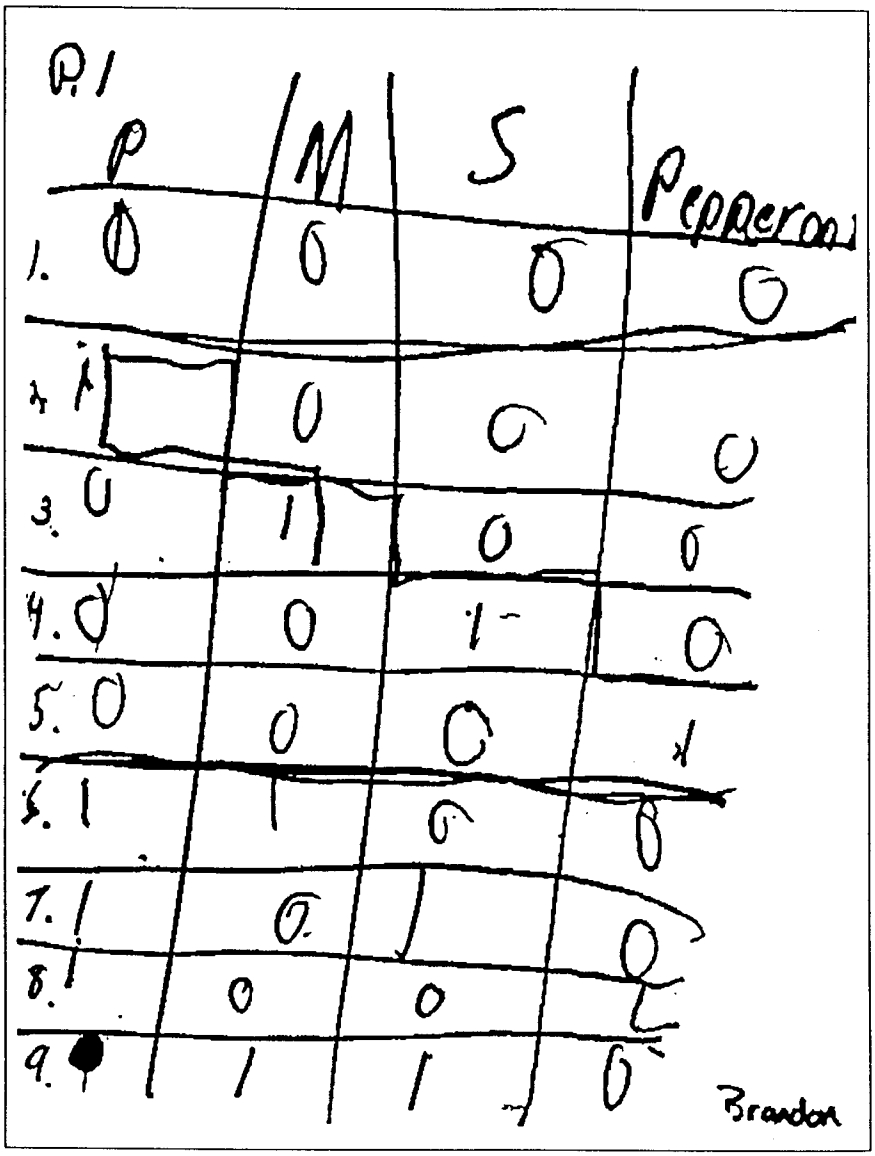


FIG. 8a

P2 / P	M	S	Pepperoni
10	1	0	1
11	0	0	1
12. 1	1	1	0
13. 1	1	0	1
14. 1	0	1	1
15 1	1		
15.	1	1	1
16. 1	1	1	1

Brandon

FIG. 8b

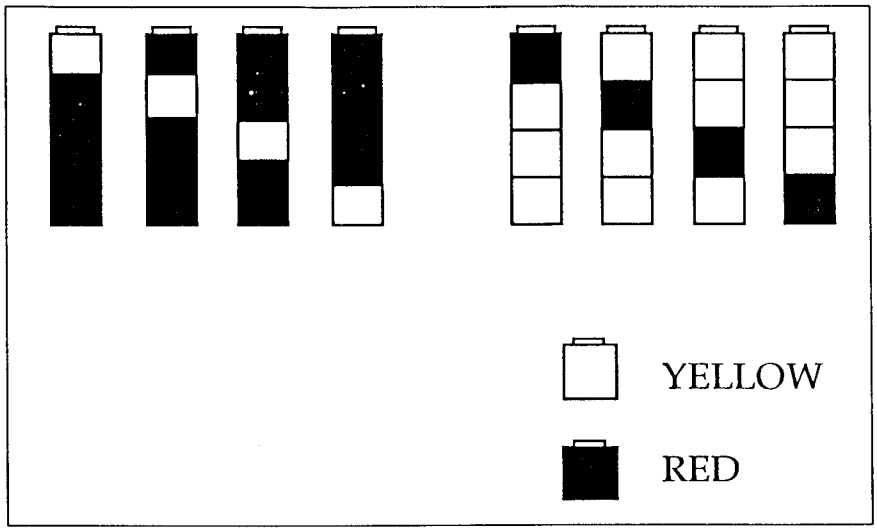


FIG. 9

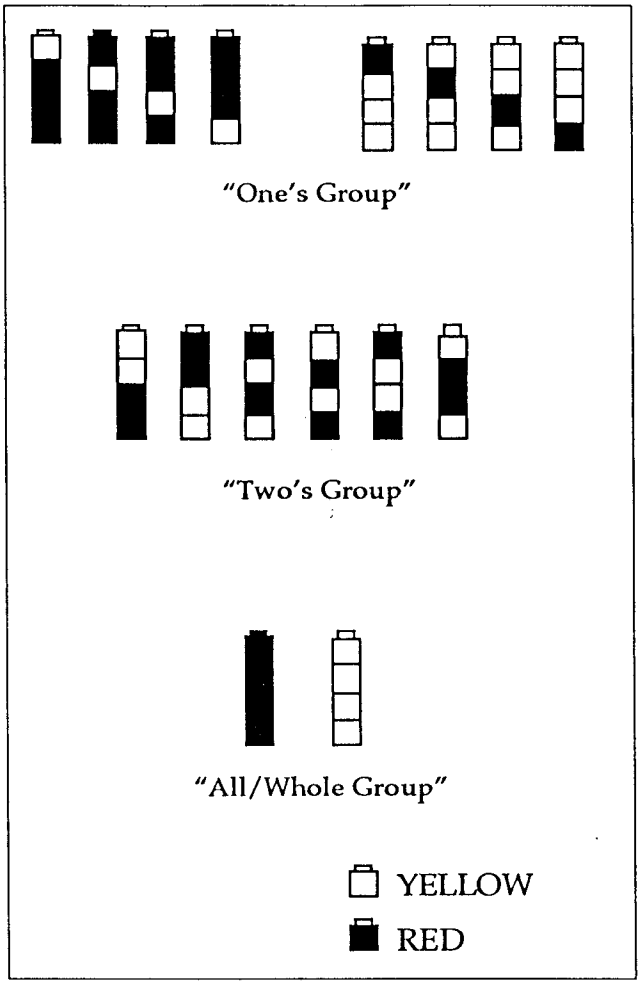


FIG. 10

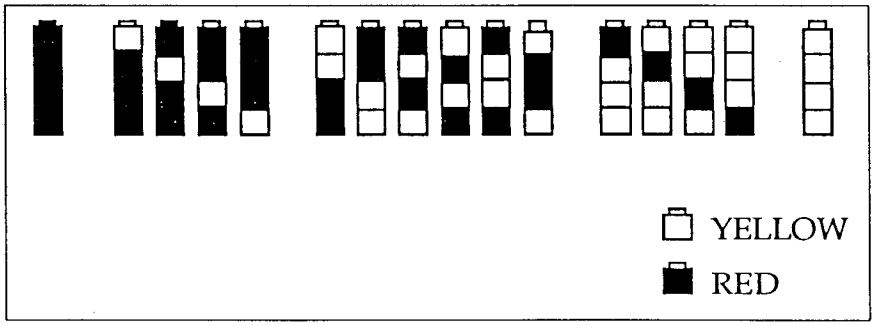


FIG. 11