Offset Spheroidal Mirrors for Gaussian Beam Optics in ZEMAX

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This memorandum discusses how to represent offset spheroidal mirrors in the ZEMAX optical design program, and how those mirrors might be used in the Gaussian beam approximation. ZEMAX is a commercial program for optical engineering and ZEMAX is a trademark of the ZEMAX Development Corporation. Mirrors in the form of a section of a prolate spheroid are often used in long-wavelength optics.

Consider three points in space: The two focii of a prolate spheroid, \( F_1, F_2 \), and a point on the surface \( C \), where the central ray strikes the surface and is reflected. The central ray travels from \( F_1 \) to \( C \), and then to \( F_2 \). Without loss of generality, we can assume these three points are in the \( \hat{x}-\hat{z} \) plane of a Cartesian coordinate system.

We will define four coordinate systems. In \((x'', y'', z'')\), the equation of the prolate \((a > b)\) spheroid is

\[
\frac{x''^2}{a^2} + \frac{y''^2}{b^2} + \frac{z''^2}{b^2} = 1 ; \tag{1}
\]

the semi-major axis of the spheroid is \(a\), the semi-minor axis is \(b\), the eccentricity is \(e = \sqrt{1 - (b/a)^2}\), and the focal distance \(f_0 = \sqrt{a^2 - b^2} = ae\) is half the length of the line segment \(F_1F_2\).

In \((x', y', z')\), we first translate to put the origin at \(F_1\), and then rotate by angle \(\theta_1\) to place point \(C\) on the positive \(\hat{z}'\) axis:

\[
\begin{pmatrix}
x' \\
y' \\
z'
\end{pmatrix} =
\begin{pmatrix}
\cos \theta_1 & 0 & \sin \theta_1 \\
0 & 1 & 0 \\
-\sin \theta_1 & 0 & \cos \theta_1
\end{pmatrix}
\begin{pmatrix}
x'' + f_0 \\
y'' \\
z''
\end{pmatrix}, \tag{2}
\]

and the inverse transform is

\[
\begin{pmatrix}
x'' \\
y'' \\
z''
\end{pmatrix} =
\begin{pmatrix}
\cos \theta_1 & 0 & -\sin \theta_1 \\
0 & 1 & 0 \\
\sin \theta_1 & 0 & \cos \theta_1
\end{pmatrix}
\begin{pmatrix}
x' \\
y' \\
z'
\end{pmatrix} +
\begin{pmatrix}
f_0 \\
0 \\
0
\end{pmatrix}. \tag{3}
\]

Substituting Equation 3 into Equation 1 yields the equation of the spheroid in the \((x', y', z')\) system:

\[
\frac{(x' \cos \theta_1 - z' \sin \theta_1 - f_0)^2}{a^2} + \frac{y'^2}{b^2} + \frac{(x' \sin \theta_1 + z' \cos \theta_1)^2}{b^2} = 1. \tag{4}
\]
Setting \( x' = y' = 0 \), and solving the resulting quadratic for \( z' \) gives the distance, \( f_1 \), between \( F_1 \) and \( C \):

\[
f_1 = \frac{b^2}{a} \left( 1 + e \sin \theta_1 \right)^{-1}.
\]

We can solve for \( \theta_1 \):

\[
\sin \theta_1 = \left( \frac{b^2}{af_1} - 1 \right) e^{-1}.
\]

Let \( f_2 \) be the distance from \( C \) to \( F_2 \). From the properties of ellipses, \( a = \frac{1}{2}(f_1 + f_2) \), \( b = \sqrt{\frac{1}{4}(f_1 + f_2)^2 - f_0^2} \), and we can express \( \theta_1 \) in terms of \( f_0 \), \( f_1 \), and \( f_2 \) only:

\[
\sin \theta_1 = \frac{f_2^2 - f_1^2 - 4f_0^2}{4f_0f_1}.
\]

This equation can also be derived by applying the law of cosines to the triangle \( F_1 CF_2 \).

In the \((x',y',z')\) system, the coordinates of point \( C \) are \((0,0,f_1)\). Substituting into equation 3, the coordinates of \( C \) in \((x'',y'',z'')\) = \((-f_0 \sin \theta_1 + f_0, 0, f_0 \cos \theta_1)\). Solve equation 1 for \( z'' \) as a function of \( x'' \), and differentiate to obtain the slope of the ellipse at point \( C \):

\[
\tan \theta_2 = \frac{dz''}{dx''} = -\frac{b}{a^2} x'' \left( 1 - \frac{x''^2}{a^2} \right)^{-1/2} = \frac{b}{a^2} (f_1 \sin \theta_1 + f_0) \left[ 1 - \left( \frac{f_1 \sin \theta_1 + f_0}{a^2} \right)^2 \right]^{-1/2}.
\]

Let \( i = \theta_2 - \theta_1 \); by an appropriate choice of coordinate, \( i \) is always between 0 and 90 degrees. Now we can define the \((x,y,z)\) system, that has its origin in the center of the mirror, the \( \hat{x}-\hat{y} \) plane tangent to the spheroid, and the \( \hat{z} \) axis pointing into the mirror surface.

\[
\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos i & 0 & \sin i \\ 0 & 1 & 0 \\ -\sin i & 0 & \cos i \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' - f_1 \end{pmatrix},
\]

and the inverse transform is

\[
\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos i & 0 & -\sin i \\ 0 & 1 & 0 \\ \sin i & 0 & \cos i \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ f_1 \end{pmatrix}.
\]

Note that \( i \) is the angle of incidence at the mirror, since the incident ray lies along the \( \hat{z}' \) axis, and \( \hat{z} \) is normal to the surface. Applying the law of cosines to the triangle \( F_1 CF_2 \), we have:

\[
4f_0^2 = f_1^2 + f_2^2 - 2f_1f_2 \cos (2i).
\]
We can now eliminate \( a, b, e, f_0, \theta_1, \) and \( \theta_2 \) in favor of \( f_1, f_2 \) and \( i \) in all of the preceding equations. The shape of the mirror can therefore be described in terms of the two focal distances and the angle of incidence. Combine equation 10 with equation 3, substitute into equation 1, and simplify:

\[
px^2 + (qx + 2r)z + y^2 + sx^2 = 0,
\]

where

\[
\begin{align*}
p &= 1 - e^2 \sin^2 \theta_2 \\
q &= e^2 \sin (2 \theta_2) \\
r &= f_1 (\cos i + e \sin \theta_2) = f_2 (\cos i - e \sin \theta_2) \\
s &= 1 - e^2 \cos^2 \theta_2
\end{align*}
\]

and

\[
\begin{align*}
f_0 &= \frac{1}{2} \sqrt{f_1^2 + f_2^2 - 2f_1f_2 \cos (2i)} \\
\theta_2 &= i + \arcsin \left( \frac{f_2^2 - f_1^2 - 4f_0^2}{4f_0f_1} \right) \\
e &= \frac{2f_0}{f_1 + f_2}
\end{align*}
\]

yielding the equation of the mirror surface sag in \((x, y, z)\) coordinates:

\[
z = \frac{1}{2p} \left[ \sqrt{(2r + qx)^2 - 4p(y^2 + sx^2)} - (2r + qx) \right].
\]

The value of \( r \) has dimensions of length, and it is always positive (either of the expressions in equation 15 can be used, depending on the sign of \( \theta_2 \)). It is the radius of curvature in the \( \hat{y}-\hat{z} \) plane, but it is not the radius of curvature seen by the beam, as will be shown in equation 21. The values of \( p \) and \( s \) are dimensionless and always between 0 and 1. The value of \( q \) is dimensionless and always between \(-1\) and 1; its sign is opposite that of the \( x'' \) coordinate of the point \( C \). Note that \( z \) is everywhere negative, since \( p \) and \( s \) are positive in equation 20. The \( \hat{z} \) direction points into the mirror, and the \( \hat{x}-\hat{y} \) plane is tangent to the mirror at point \( C \), the origin of the \((x, y, z)\) system. The direction of \( \hat{x} \) is such that its dot product with the vector from \( F_1 \) to \( F_2 \) is positive.

Since we know the two focal distances \( f_1 \) and \( f_2 \), the thin lens formula gives the paraxial focal length \( f \):

\[
\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} = \frac{2 \cos i}{r},
\]

using equation 15 to substitute for \( f_1 \) and \( f_2 \). So \( f = r/(2 \cos i) \), and the central radius of curvature is \( r/\cos i \).
Coefficients of Extended Polynomial

<table>
<thead>
<tr>
<th>Term</th>
<th>Coefficient/2r</th>
</tr>
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<tbody>
<tr>
<td>$x^2$</td>
<td>$1 - s$</td>
</tr>
<tr>
<td>$x^3$</td>
<td>$qs$</td>
</tr>
<tr>
<td>$x^4$</td>
<td>$q$</td>
</tr>
<tr>
<td>$x^5$</td>
<td>$p - ps^2 - q^2s$</td>
</tr>
<tr>
<td>$x^6$</td>
<td>$2p - 2ps - q^2$</td>
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<td>$x^7$</td>
<td>$3pq^2s^2 + q^3s$</td>
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<tr>
<td>$x^8$</td>
<td>$6pq^2s + q^3$</td>
</tr>
<tr>
<td>$x^9$</td>
<td>$3pq$</td>
</tr>
<tr>
<td>$x^{10}$</td>
<td>$2p^2 - 2p^2s^3 - 6pq^2s^2 - q^4s$</td>
</tr>
<tr>
<td>$x^{11}$</td>
<td>$6p^2 - 6p^2s^2 - 12pq^2s - q^4$</td>
</tr>
<tr>
<td>$x^{12}$</td>
<td>$6p^2 - 6p^2s - 6pq^2$</td>
</tr>
<tr>
<td>$x^{13}$</td>
<td>$10p^2qs^3 + 10pq^3s^2 + q^5s$</td>
</tr>
<tr>
<td>$x^{14}$</td>
<td>$30p^2qs^2 + 20pq^3s + q^5$</td>
</tr>
<tr>
<td>$x^{15}$</td>
<td>$30p^2qs + 10pq^3$</td>
</tr>
<tr>
<td>$x^{16}$</td>
<td>$10p^2q$</td>
</tr>
<tr>
<td>$x^{17}$</td>
<td>$5p^3 - 5p^3s^4 - 30p^2q^2s^3 - 15pq^4s^2 - q^6s$</td>
</tr>
<tr>
<td>$x^{18}$</td>
<td>$20p^3 - 20p^3s^3 - 90p^2q^2s^2 - 30pq^4s - q^6$</td>
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<tr>
<td>$x^{19}$</td>
<td>$30p^3 - 30p^3s^2 - 90p^2q^2s - 15pq^4$</td>
</tr>
<tr>
<td>$x^{20}$</td>
<td>$20p^3 - 20p^3s - 30p^2q^2$</td>
</tr>
</tbody>
</table>

One way to represent this surface in ZEMAX is as an “Extended Polynomial” surface. This is a standard, radially-symmetric conic surface with the addition of polynomial terms of the form $A(x/g)^m(y/g)^n$, where $A$ is a coefficient, $g$ is a normalization, and $m$ and $n$ range over the positive integers. The appropriate parameters of the ZEMAX surface can be obtained by subtracting a standard conic surface from equation 20, and expanding the residual in a Taylor expansion in $x$ and $y$. Take the radius of curvature to be $-r$, and the conic constant to be $k = p - 1$. The value of $k$ will always be between $-1$ and $0$, so the standard surface is always a prolate spheroid. The coefficients, $A$, of the polynomial terms are given in the table above. The normalization value for $x$ and $y$ is $g = +2r$, and each entry in the table above should be multiplied by $2r$ to produce the ZEMAX coefficient. Convergence of the expansion is assured, as long as $|x| < 2r$ and $|y| < 2r$. Note that there are terms containing odd powers of $x$, but by symmetry no terms containing odd powers of $y$. Odd powers of $q$ appear in all odd terms in $x$, so reversing the sign of $q$ is equivalent to reversing the direction of $\hat{x}$. In the plane $x = 0$, all terms in the polynomial vanish, and the surface is perfectly fit by the standard surface with curvature $-1/r$ and conic constant $k$. Note that as discussed above, the paraxial curvature is $(-\cos i)/r$. The entire surface must


be preceded by a coordinate break creating a rotation of angle $-i$ about \( \hat{y} \), and the surface
is followed by another coordinate break also of angle $-i$ about \( \hat{y} \).

Figure 1 shows an example of this representation of an offset mirror in ZEMAX. The
model is not perfect, due to round-off in the ZEMAX spreadsheets and the finite number
of terms. This type of representation is useful if the mirror shape is to be perturbed in
an optimization analysis, for example to improve the field of view at the expense of image
quality in the center of the field.

An alternative representation of a section of an offset prolate spheroid in ZEMAX is
the “Conjugate” surface. The Conjugate surface can provide an exact representation of a
prolate spheroid, but no perturbations of that surface are allowed.

The Conjugate surface is defined by the equation:

\[
\sqrt{(x'' - x_1)^2 + (y'' - y_1)^2 + (z'' - z_1)^2} + \sqrt{(x'' - x_2)^2 + (y'' - y_2)^2 + (z'' - z_2)^2} = f_1 + f_2,
\]

where \( f_1 = \sqrt{x_1^2 + y_1^2 + z_1^2} \) and \( f_2 = \sqrt{x_2^2 + y_2^2 + z_2^2} \). Here \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\)
are the coordinates of \( F_1 \) and \( F_2 \). Then

\[
f_0 = \frac{1}{2} \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.
\]

Note that the origin of the \((x'', y'', z'')\) coordinate system lies in the surface of the prolate
spheroid and ZEMAX assumes that this point is the intersection, \( C \), of the surface with the
central ray. The angle of incidence, \( i \), at the point of reflection is given by the dot product
of the two vectors from the origin to the to focii:

\[
i = \frac{1}{2} \arccos \left( \frac{x_1 x_2 + y_1 y_2 + z_1 z_2}{f_1 f_2} \right).
\]

Since we know \( f_1 \), \( f_2 \), and \( i \), we can derive all the constants in equation 12. Like the
\((x, y, z)\) coordinate system, the \((x'', y'', z'')\) coordinate system has its origin at \( C \), but the
two coordinate systems are connected by (potentially arbitrary) rotation. The unit vector \( \hat{z} \) is normal to the surface and points out of the surface, away from the focii. Consider two
normalized vectors that point from \( C \) toward \( F_1 \) and \( F_2 \). The direction of \( \hat{z} \) is opposite to
the average of these two vectors, and \( \hat{z} \) itself is \(-1\) times the normalized average of those
vectors, so its components in the \((x'', y'', z'')\) coordinate system are:

\[
\hat{z} = -\frac{(x_1/f_1 + x_2/f_2, y_1/f_1 + y_2/f_2, z_1/f_1 + z_2/f_2)}{\sqrt{(x_1/f_1 + x_2/f_2)^2 + (y_1/f_1 + y_2/f_2)^2 + (z_1/f_1 + z_2/f_2)^2}}
\]

the unit vector \( \hat{y} \) is the cross product of those two normalized vectors:

\[
\hat{y} = \frac{1}{f_1 f_2 \sin(2i)} \left( y_1 z_2 - y_2 z_1, z_1 x_2 - z_2 x_1, x_1 y_2 - x_2 y_1 \right),
\]
and of course
\[ \hat{x} = \hat{y} \times \hat{z}. \quad (27) \]

The components of these vectors are the components of the arbitrary 3-d rotation matrix connecting the \((x, y, z)\) coordinate system with the \((x'', y'', z'')\) coordinate system. A frequently-encountered special case has \(F_1, F_2,\) and \(C\) in the \(y-z\) plane so \(x_1 = x_2 = 0,\) and in that case the relation between the \((x, y, z)\) coordinate system and the \((x'', y'', z'')\) coordinate system is simply a rotation about \(\hat{x}''\) by an angle:
\[ \phi = \pm i + \text{atan2}(z_2, y_2) - 180^\circ \quad (28) \]

followed by a rotation of \(90^\circ\) around \(\hat{z}:\)
\[
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix}
= \begin{pmatrix}
  0 & -\cos \phi & \sin \phi \\
  1 & 0 & 0 \\
  0 & \sin \phi & \cos \phi
\end{pmatrix}
\begin{pmatrix}
  x'' \\
  y'' \\
  z''
\end{pmatrix}. \quad (29)
\]

Equation 22 is an alternate representation of the surface in equation 20, and equations 22 through 27 connect the two coordinate systems used.

As with the Extended Polynomial surface, the Conjugate surface must be both preceded and followed by a coordinate break rotation around \(\hat{y}''\) by an angle \(-i.\) Figure 2 shows the same spheroid as in Figure 1, but represented as a Conjugate surface. Note that the focus is perfect in the center of the field, unlike the 6th-order extended polynomial.

Offset spheroidal mirrors are often used in a context where diffraction effects are significant, because the scale of the optics is not very large compared to a wavelength. We use the Gaussian beam approximation, and introduce some new variables:

\[
\begin{align*}
\lambda &= \text{wavelength} \\
\lambda_1 &= \text{distance from input beamwaist to mirror center} \\
\lambda_2 &= \text{distance from output beamwaist to mirror center} \\
\omega_1 &= \text{radius of input beamwaist} \\
\omega_2 &= \text{radius of output beamwaist} \\
\omega_1 &= \text{Rayleigh range of input beamwaist} \\
\omega_2 &= \text{Rayleigh range of output beamwaist}
\end{align*}
\]

The beamwaists are locations where the curvature of the wavefront is zero. The radii of the beamwaists, \(\omega_1\) and \(\omega_2,\) characterize the Gaussian distribution of power in the plane of the beamwaist, and the Rayleigh ranges \(\omega_1\) and \(\omega_2\) characterize the Lorentzian profile of intensity along the axis—the “far field” is a distance greater than the Rayleigh range from
the beamwaist. In the far field, the half-angle of the beam divergence is \( \lambda / (\pi w) \). A useful paper describing the relation between these variables is S. A. Self, *Applied Optics*, Vol. 22, No. 5, p.p. 658-661. The Rayleigh ranges are simple functions of the waist radii:

\[
\begin{align*}
  z_1 &= \frac{\pi w_1^2}{\lambda} \\
  z_2 &= \frac{\pi w_2^2}{\lambda}.
\end{align*}
\]

The ratio of the size of the output and input beam waists is given by the magnification:

\[
m = \frac{w_2}{w_1} = \frac{f}{\sqrt{(d_1 - f)^2 + z_1^2}},
\]

where \( f \) is the paraxial focal length given by equation 21. The Gaussian beam equivalent of the thin-lens formula then becomes:

\[
\frac{d_1 - f}{d_1 (d_1 - f) + z_1^2} + \frac{1}{d_2} = \frac{1}{f},
\]

or

\[
\frac{1}{d_1} + \frac{d_2 - f}{d_2 (d_2 - f) + z_2^2} = \frac{1}{f}.
\]

Note that in the case of \( z_1 \) and \( z_2 \) negligible compared to \( f \), equations 32, 33, and 34 revert to their thin-lens equivalents.

Given an input beamwaist radius, \( w_1 \), at a distance \( d_1 \) from the center of a mirror described by equations 1 through 21 (giving a value for \( f \)), with an angle of incidence \( i \), equations 30, 32, and 33 allow us to solve for the output beamwaist \( w_2 \) and its distance from the mirror center, \( d_2 \). These values, however, are a function of wavelength, \( \lambda \). Also, there are an infinite number of possible values of \( f_1 \) and \( f_2 \), all of which result in the same value of \( f \).

An optical design will usually relay the beam through a series of mirrors. We’d like location of the final focus to be wavelength independent, and we want to minimize aberrations over a small image around the central beam. These criteria can be used to fully determine the optical system.

Note that in equation 33, the only term that is a function of wavelength is \( z_1 \). If the input distance \( d_1 \) is set equal to \( f \), the term containing \( z_1 \) vanishes, and then \( d_2 = f \), independent of wavelength. If we have a mirror whose paraxial focal length is \( f \), and we place the input beamwaist at \( d_1 = f \), then the output beamwaist will be at \( d_2 = f \) as well, for all wavelengths. Consider, however, what this means in the short wavelength (ray trace) limit — if there is an image located a distance \( f \) from the input side of the optic, the beam on the other side is collimated. In the Gaussian beam approximation the “collimated”
beam has a beamwaist a distance \( f \) on the other side. It is important to keep track of which beamwaist is at an image, and which beamwaist is a collimated beam.

Two mirrors can be combined into a “Gaussian telescope”, where an image at the input focus to one mirror is transferred to another image at the output focus of the other mirror. Between the two mirrors, there is a beamwaist corresponding to a collimated beam, and the output beamwaist size of the first mirror matches the input beamwaist of the second mirror. Call the first mirror “\( A \)” and the second mirror “\( B \)”. Mirror \( A \) is a distance \( f_A \) from the first image. Mirror \( B \) is located \( f_A + f_B \) from mirror \( A \), and the second image is located a distance \( f_B \) from mirror \( B \).

The two focal lengths of the two prolate spheroids (\( f_{1A}, f_{2A}, f_{1B}, \) and \( f_{2B} \)), can be set by the following criterion: the phase center of the beam from the image is located at a focus of the spheroid. In a Gaussian beam, the wavefront far from a beamwaist is spherical, but the center of the sphere is not located at the beamwaist, it is located beyond the beamwaist by a factor \((1 + z_R^2/d^2)\). Specifically, if the beamwaist of a Gaussian beam is located at \( z = 0 \), the radius of curvature of the wavefronts is

\[
R(z) = z \left[ 1 + \left( \frac{z_R}{z} \right)^2 \right],
\]

where \( z_R \) is the Rayleigh range. If we let

\[
f_{1A} = d_{1A} \left[ 1 + \left( \frac{z_{1A}}{d_{1A}} \right)^2 \right] = f_A \left[ 1 + \left( \frac{z_{1A}}{f_A} \right)^2 \right],
\]

then the shape of mirror \( A \) and the curvature of the wavefronts of wavelength \( \lambda \) will be the same as a mirror with waves in the geometric optics limit of short wavelength: when the wavefront arrives at mirror \( A \), the center of curvature of that wavefront will be located at the \( F_1 \) focus of the spheroid. Equations 36 and 21 then determine

\[
f_{2A} = f_A \left[ 1 + \left( \frac{f_A}{z_{1A}} \right)^2 \right].
\]

The output focus length of mirror \( B \) is similarly constrained by the location of the phase center near the output image:

\[
f_{2B} = d_{2B} \left[ 1 + \left( \frac{z_{2B}}{d_{2B}} \right)^2 \right] = f_B \left[ 1 + \left( \frac{z_{2B}}{f_B} \right)^2 \right],
\]

and equations 38 and 21 then determine

\[
f_{1B} = f_B \left[ 1 + \left( \frac{f_B}{z_{2B}} \right)^2 \right].
\]
These equations fully constrain all the parameters of the Gaussian telescope, if the two paraxial focal lengths $f_A$ and $f_B$, and the two angles of incidence, $i_A$ and $i_B$ are freely chosen. If $f_A = f_B$ and $i_A = i_B$, then the magnification of the Gaussian telescope is unity, and aberrations cancel by symmetry. If $f_A \neq f_B$, then the condition of minimum aberrations places a different constraint on the relation between $i_A$ and $i_B$. See, for example C. Dragone, “A first-order treatment of aberrations in Cassegrainian and Gregorian antennas,” IEEE Trans. Antennas and Propagation AP-30, p. 331, 1982.

We can realize a Gaussian telescope in ZEMAX. Figure 3 shows a symmetric Gaussian telescope with arbitrary values: $f = 20 \text{ mm}$ and $i = 37.3 \text{ deg}$. The wavelength, $\lambda = 10 \text{ mm}$ has been chosen to be large compared to the focal length. The Gaussian beam is launched into the telescope with a “Paraxial” element with a focal length of 100 mm. In the setup, this lens is assumed to have an input beamwaist 100 mm in front of it, resulting in an image beamwaist 100 mm behind it. We arbitrarily set the converging half-angle of this beam to $\frac{\pi}{6}$ radians, so $w_{1A} = \frac{6\lambda}{\pi^2} = 6.079 \text{ mm}$, and $z_{1A} = \frac{\pi w_{1A}}{\lambda} = 11.61 \text{ mm}$. Then equation 35 gives $f_{1A} = 26.739 \text{ mm}$, equation 36 gives $f_{2A} = 79.35 \text{ mm}$, and equations 23 - 28 give the values for the “Conjugate” surfaces. Note the sign reversals that come with reflection from Mirror A.

The ZEMAX Skew Gaussian Beam analysis shown in Figure 4 is essentially correct. The Paraxial Gaussian Beam analysis is off by several millimeters.
Fig. 1.— Screenshot of an offset prolate spheroidal mirror, represented as an Extended Polynomial in ZEMAX. The mirror parameters were chosen at random.
Fig. 2.— Screenshot of the prolate spheroidal mirror, represented as a Conjugate surface.
Fig. 3.— Realization of a Gaussian telescope in ZEMAX.
Fig. 4.— Skew Gaussian Beam analysis of the Gaussian telescope in Figure 3.

<table>
<thead>
<tr>
<th>Sur</th>
<th>Size</th>
<th>Waist</th>
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<th>Radius</th>
<th>Divergence</th>
<th>Rayleigh</th>
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