Ideal Magnetohydrodynamics (MHD)

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These lecture notes are largely based on Lectures in Magnetohydrodynamics by the late Dalton Schnack, Ideal Magnetohydrodynamics by Jeffrey Freidberg, Plasma Physics for Astrophysics by Russell Kulsrud, Magnetic Reconnection by Eric Priest and Terry Forbes, course notes from similar classes taught by Ellen Zweibel and Chris Hegna, and a picture of flying wombats that I found on the internet.
Outline

- Introducing ourselves (since we forgot to on the first day!)
- Discuss location of course
- Overview of MHD
  - Approximation
  - Usefulness
  - Applications
- The equations of MHD and their physical meaning
  - Continuity equation
  - Momentum equation
  - Energy equation
  - Faraday’s law
  - Ohm’s law
What is MHD?

MHD couples Maxwell’s equations with hydrodynamics to describe the macroscopic behavior of highly conducting fluids such as plasmas
Ideal MHD at a glance (cgs units)

Continuity Equation
\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0 \]

Momentum Equation
\[ \rho \left( \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \mathbf{V} = \frac{\mathbf{J} \times \mathbf{B}}{c} - \nabla p \]

Ampere’s law
\[ \mathbf{J} = \frac{c}{4\pi} \nabla \times \mathbf{B} \]

Faraday’s law
\[ \frac{\partial \mathbf{B}}{\partial t} = -c \nabla \times \mathbf{E} \]

Ideal Ohm’s law
\[ \mathbf{E} + \frac{\mathbf{V} \times \mathbf{B}}{c} = 0 \]

Divergence constraint
\[ \nabla \cdot \mathbf{B} = 0 \]

Adiabatic Energy Equation
\[ \frac{d}{dt} \left( \frac{p}{\rho^\gamma} \right) = 0 \]

Definitions: \( \mathbf{B} \), magnetic field; \( \mathbf{V} \), plasma velocity; \( \mathbf{J} \), current density; \( \mathbf{E} \), electric field; \( \rho \), mass density; \( p \), plasma pressure; \( \gamma \), ratio of specific heats (usually 5/3); \( t \), time.
MHD is a low-frequency, long-wavelength approximation

- MHD is valid on time scales longer than the inverses of the plasma frequencies and cyclotron frequencies for both ions and electrons:

$$\tau \gg \omega_{pe}^{-1}, \omega_{pi}^{-1}, \Omega_{ce}^{-1}, \Omega_{ci}^{-1}$$  \hspace{1cm} (1)

- MHD is valid on length scales longer than the Debye length and electron/ion gyroradii:

$$L \gg \lambda_D, r_{Le}, r_{Li}$$  \hspace{1cm} (2)

- MHD assumes quasineutrality (since \(L \gg \lambda_D\))
MHD is a low-frequency, long-wavelength approximation

- MHD assumes that collisions are frequent enough for the particle distribution function to be Maxwellian with $T_i = T_e$
- Ideal MHD assumes an adiabatic equation of state
  - No additional heating, cooling, or dissipation
- MHD assumes that the plasma is fully ionized
- MHD ignores the most important advances in physics since $\sim 1860$
  - Ignore relativity (assume $V^2 \ll c^2$)
  - Ignore quantum mechanics
  - Ignore displacement current in Ampere’s law (assume $V^2 \ll c^2$)
When is MHD useful?

- MHD traditionally describes macroscopic force balance, equilibria, and dynamics
- MHD is a good predictor of stability
  - The most catastrophic instabilities are unstable in ideal MHD
  - Important in laboratory plasmas, solar atmosphere, etc.
- Systems that are often described using MHD include:
  - Solar wind, heliosphere, and Earth’s magnetosphere\(^1\)
  - Inertial range of plasma turbulence
  - Neutron star magnetospheres
- MHD is a reasonably good approximation in many astrophysical plasmas
  - However, extensions are often needed

\(^1\)On large scales!
When is MHD not useful?

- MHD has limited applicability when:
  - Non-fluid or kinetic effects are important
    - Dissipation in the turbulent solar wind
    - Magnetic reconnection
    - Small-scale dynamics in Earth’s magnetosphere
  - Particle distribution functions are non-Maxwellian
    - Cosmic rays
  - The plasma is weakly ionized
    - Solar photosphere/chromosphere, molecular clouds, protoplanetary disks, Earth’s ionosphere, some laboratory plasmas

- MHD is mediocre at describing the dynamics of laboratory plasmas but remains a good predictor of stability
Deriving the continuity equation

- Pick a closed volume \( \mathcal{V} \) bounded by a fixed surface \( S \) containing plasma with mass density \( \rho \)
- The total mass contained in the volume is
  \[
  M = \int_{\mathcal{V}} \rho \, \mathrm{d}V
  \]  
  (3)
- The time derivative of the mass in \( \mathcal{V} \) is
  \[
  \frac{\mathrm{d}M}{\mathrm{d}t} = \int_{\mathcal{V}} \frac{\partial \rho}{\partial t} \, \mathrm{d}V
  \]  
  (4)
The continuity equation describes conservation of mass

- The mass flowing through a surface element \( dS = \hat{n}dS \) is \( \rho \mathbf{V} \cdot dS \), where the unit vector \( \hat{n} \) is pointing outward.
- The integral of \( \rho \mathbf{V} \cdot dS \) must equal \( -\frac{dM}{dt} \):

\[
\int_{\mathcal{V}} \frac{\partial \rho}{\partial t} \, d\mathcal{V} = -\int_{\mathcal{S}} \rho \mathbf{V} \cdot d\mathbf{S}
\]  

(5)

This says that the change in mass inside \( \mathcal{V} \) equals the mass entering or leaving the surface.
- Using Gauss’ theorem we arrive at

\[
\int_{\mathcal{V}} \left[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) \right] \, d\mathcal{V} = 0
\]  

(6)

- This must be true for all possible volumes so the integrand must equal zero.
The continuity equation in conservative form is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0 \quad (7)$$

Conservative form is usually given by

$$\frac{\partial}{\partial t} \text{(stuff)} + \nabla \cdot \text{(flux of stuff)} = 0 \quad (8)$$

Source and sink terms go on the RHS

Example: In a partially ionized plasma, there are continuity equations for both the ions and neutrals. Ionization acts as a source term in the ion continuity equation and a sink term in the neutral continuity equation.

The mass flux is given by $\rho \mathbf{V}$
The second golden rule of astrophysics

The density of wombats times the velocity of wombats gives the flux of wombats.
The continuity equation

- Using vector identities we may write the continuity equation as

\[
\frac{\partial \rho}{\partial t} + \mathbf{V} \cdot \nabla \rho = -\rho \nabla \cdot \mathbf{V} \tag{9}
\]

- The advective derivative \( \mathbf{V} \cdot \nabla \rho \) is a directional derivative that measures the change of \( \rho \) in the direction of \( \mathbf{V} \)

- The compression term
  - \( \nabla \cdot \mathbf{V} < 0 \iff \text{converging flow} \iff \text{compression} \)
  - \( \nabla \cdot \mathbf{V} > 0 \iff \text{diverging flow} \iff \text{dilation} \)
  - \( \nabla \cdot \mathbf{V} \equiv 0 \iff \text{the plasma is incompressible} \)
The advective derivative $\mathbf{V} \cdot \nabla$ is used to describe the spatial variation of a field in the direction of the flow.

For a scalar quantity $\varphi$, the advective derivative is given by

$$\mathbf{V} \cdot \nabla \varphi = V_x \frac{\partial \varphi}{\partial x} + V_y \frac{\partial \varphi}{\partial y} + V_z \frac{\partial \varphi}{\partial z},$$

which is also a scalar.
For a vector field \( \mathbf{F} \), the advective derivative may be treated as either \((\mathbf{V} \cdot \nabla) \mathbf{F}\) or as the tensor derivative \(\mathbf{V} \cdot (\nabla \mathbf{F})\). Both forms are equivalent, but \((\mathbf{V} \cdot \nabla) \mathbf{F}\) is easier to work with.

\[
(\mathbf{V} \cdot \nabla) \mathbf{F} = \left[ \begin{array}{c} V_x \\ V_y \\ V_z \end{array} \right] \cdot \left[ \begin{array}{c} \partial_x \\ \partial_y \\ \partial_z \end{array} \right] \mathbf{F}
\]

\[
= (V_x \partial_x + V_y \partial_y + V_z \partial_z) \left( \begin{array}{c} F_x \\ F_y \\ F_z \end{array} \right)
\]

\[
= \left( \begin{array}{c} V_x \partial_x F_x + V_y \partial_y F_x + V_z \partial_z F_x \\ V_x \partial_x F_y + V_y \partial_y F_y + V_z \partial_z F_y \\ V_x \partial_x F_z + V_y \partial_y F_z + V_z \partial_z F_z \end{array} \right) \quad (11)
\]

The order of operations does not matter so we write \(\mathbf{V} \cdot \nabla \mathbf{F}\).

See the *NRL Plasma Formulary* for curvilinear coordinates.
The Eulerian and Lagrangian forms of the continuity equation are equivalent

- The Eulerian form follows the density at a fixed location in space:

  \[ \frac{\partial \rho}{\partial t} + \mathbf{V} \cdot \nabla \rho = -\rho \nabla \cdot \mathbf{V} \]  
  \[ (12) \]

- The Lagrangian form allows us to follow a volume element that is co-moving with the fluid:

  \[ \frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{V} = 0 \]  
  \[ (13) \]

where the total derivative is given by

\[ \frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \]  
\[ (14) \]

and measures the change of a quantity as we move with the fluid. The advective derivative links the Eulerian and Lagrangian forms.
The momentum equation is derived from Newton’s 2nd law

- Newton’s second law of motion for a fluid element is

\[ \rho \frac{d\mathbf{V}}{dt} = \mathbf{F} \]  

where \( \mathbf{F} \) is the force per unit volume.

- Example forces include:
  - Lorentz force: \( \mathbf{F}_L = \frac{\mathbf{J} \times \mathbf{B}}{c} \)
  - Pressure gradient force: \( \mathbf{F}_p = -\nabla p \)
  - Gravity: \( \mathbf{F}_g = -\rho \mathbf{g} \) or \( \mathbf{F}_g = -\nabla \phi \) for gravitational potential \( \phi \)
  - Viscosity: \( \mathbf{F}_V = \nabla \cdot \mathbf{\Pi} \), where \( \mathbf{\Pi} \) is the viscous stress tensor.

- The ideal MHD momentum equation in Eulerian form is

\[ \rho \left( \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \mathbf{V} = \frac{\mathbf{J} \times \mathbf{B}}{c} - \nabla p \]  

where we neglect gravity and ignore viscous forces.
The pressure gradient force $-\nabla p$ pushes plasma from regions of high plasma pressure to low plasma pressure.

- The pressure gradient force is orthogonal to isobars.
- This is the Restoring force for sound waves.
- Resulting motions are not necessarily in the direction of the pressure gradient force (e.g., when other forces are acting on the fluid).
Where does the Lorentz force come from?

- The Lorentz force acting on a single particle is

\[
F = q \left( E + \frac{V \times B}{c} \right) \tag{17}
\]

- The current density is given by

\[
J = \sum_{\alpha} n_{\alpha} q_{\alpha} V_{\alpha} \tag{18}
\]

where \( \alpha \) includes all species of ions and electrons. For a quasineutral plasma with electrons and singly charged ions, this becomes

\[
J = en (V_i - V_e) \tag{19}
\]

where \( n = n_e = n_i \), \( V_i \) is the ion velocity, and \( V_e \) is the electron velocity.
The Lorentz force includes a magnetic tension force and a magnetic pressure force.

- Use Ampere’s law and vector identities to decompose the Lorentz force term into two components:

\[
\frac{J \times B}{c} = \frac{(\nabla \times B) \times B}{4\pi}
\]

\[
= \frac{B \cdot \nabla B}{4\pi} - \nabla \left( \frac{B^2}{8\pi} \right) \quad (20)
\]

- While the Lorentz force must be orthogonal to be \(B\), both of these terms may have components along \(B\). The parallel component of the above tension term cancels out the parallel part of the magnetic pressure term (Kulsrud §4.2).
The curvature vector $\kappa$ gives the rate at which the tangent vector turns.

- Define $\hat{b}$ as a unit vector in the direction of $B$: $\hat{b} \equiv \frac{B}{|B|}$
- The curvature vector $\kappa$ points toward the center of curvature and is given by
  \[ \kappa \equiv \hat{b} \cdot \nabla \hat{b} = -\frac{R}{R^2} \]  
  (21)
where $R$ is a vector from the center of curvature to the point we are considering. Note that $|\kappa| = R^{-1}$ and $\kappa \cdot \hat{b} = 0$. 
The Lorentz force can be decomposed into two terms with forces orthogonal to $\mathbf{B}$ using field line curvature.

Next use the product rule to obtain

$$\mathbf{B} \cdot \nabla \mathbf{B} = B \hat{b} \cdot \nabla (B \hat{b}) = \frac{\hat{b}(\hat{b} \cdot \nabla)B^2}{2} + B^2 \hat{b} \cdot \nabla \hat{b} \quad (22)$$

We can then write the Lorentz force as

$$\frac{\mathbf{J} \times \mathbf{B}}{c} = \kappa \frac{B^2}{4\pi} - \nabla_\perp \left( \frac{B^2}{8\pi} \right)$$

magnetic tension

magnetic pressure

where all terms are orthogonal to $\mathbf{B}$.

The operator $\nabla_\perp$ keeps only the derivatives orthogonal to $\mathbf{B}$:

$$\nabla_\perp \equiv \nabla - \hat{b}(\hat{b} \cdot \nabla) \quad (24)$$

\footnote{Note: the terms in this formulation for magnetic tension and pressure differ from the corresponding terms in Eq. 20.}
The magnetic tension force wants to straighten magnetic field lines

- The magnetic tension force is directed radially inward with respect to magnetic field line curvature
Regions of high magnetic pressure exert a force towards regions of low magnetic pressure.

- The magnetic pressure is given by $p_B \equiv \frac{B^2}{8\pi}$.
The ratio of the plasma pressure to the magnetic pressure is an important dimensionless number

- Define plasma $\beta$ as
  $$
  \beta \equiv \frac{\text{plasma pressure}}{\text{magnetic pressure}} \equiv \frac{p}{B^2 / 8\pi}
  $$

- If $\beta \ll 1$ then the magnetic field dominates
  - Solar corona
  - Poynting flux driven jets
  - Tokamaks ($\beta \lesssim 0.1$)

- If $\beta \gg 1$ then plasma pressure forces dominate
  - Stellar interiors

- If $\beta \sim 1$ then pressure/magnetic forces are both important
  - Solar chromosphere
  - Parts of the solar wind and interstellar medium
  - Some laboratory plasma experiments
The adiabatic energy equation provides the closure for ideal MHD

- The Lagrangian form of the adiabatic energy equation is
  \[
  \frac{d}{dt} \left( \frac{p}{\rho^\gamma} \right) = 0 \quad (25)
  \]

- The Eulerian form of the adiabatic energy equation is
  \[
  \left( \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) p = -\gamma \rho \nabla \cdot \mathbf{V} \quad (26)
  \]

  where the term on the RHS represents heating/cooling due to adiabatic compression/expansion.
  - The entropy of any fluid element is constant
  - Ignores thermal conduction, non-adiabatic heating/cooling

- This is generally a mediocre approximation, but is useful for some situations (e.g., MHD waves)
Faraday’s law tells us how the magnetic field varies with time

Faraday’s law is unchanged from Maxwell’s equations:

\[
\frac{\partial \mathbf{B}}{\partial t} = -c \nabla \times \mathbf{E}
\]  

(27)

But how do we get the electric field?
We get the electric field from Ohm’s law

The electric field $E'$ seen by a conductor moving with velocity $V$ is given by

$$E' = \frac{E + \frac{V \times B}{c}}{\sqrt{1 - \frac{V^2}{c^2}}} \quad (28)$$

This is Lorentz invariant, but the fluid equations are only Galilean invariant! Let’s expand the denominator.

$$E' = \left( E + \frac{V \times B}{c} \right) \left( 1 - \frac{1}{2} \frac{V^2}{c^2} + \ldots \right)$$

$$= E + \frac{V \times B}{c} + \mathcal{O} \left( \frac{V^2}{c^2} \right) \quad (29)$$

By setting $E' = 0$, we arrive at the ideal Ohm’s law

$$E + \frac{V \times B}{c} = 0 \quad (30)$$

which ignores $\mathcal{O} \left( \frac{V^2}{c^2} \right)$ terms and is Galilean invariant.
Ohm’s law can be combined with Faraday’s law for the induction equation

- Using \( \mathbf{E} + \frac{\mathbf{V} \times \mathbf{B}}{c} = 0 \) and \( \frac{\partial \mathbf{B}}{\partial t} = -c \nabla \times \mathbf{E} \), we arrive at
  \[
  \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B})
  \]
  (31) advection

- The ideal Ohm’s law neglects contributions to \( \mathbf{E} \) from resistivity, the Hall effect, electron inertia, and (in partially ionized plasmas) ambipolar diffusion

- As we will soon see, the ideal Ohm’s law leads to the magnetic field and plasma being frozen into each other so that magnetic topology is preserved

- Ideal MHD plasmas are perfectly conducting
The low-frequency Ampere’s law

- Ampere’s law without displacement current is

$$\mathbf{J} = \frac{c}{4\pi} \nabla \times \mathbf{B} \quad (32)$$

- There is no time-dependence, so we can replace \( \mathbf{J} \) in other equations using this expression.

- This formulation implies that

$$\nabla \cdot \mathbf{J} = 0 \quad (33)$$

which is a necessary condition for quasineutrality.

- MHD treats the plasma as a single fluid, but recall that \( \mathbf{J} \) also represents the relative drift between ions and electrons.

$$\mathbf{J} \equiv \sum_\alpha n_\alpha q_\alpha \mathbf{V}_\alpha \quad (34)$$
And of course, the most boringest of Maxwell’s equations must remain satisfied

- The divergence constraint, also known as Gauss’ law for magnetism. Huzzah! …

\[ \nabla \cdot \mathbf{B} = 0 \quad (35) \]

- Magnetic monopoles do not exist
- The magnetic charge density equals zero
- \( \mathbf{B} \) is a \textit{solenoidal} (divergence-free) field

- We might as well put it in integral form while we’re here…

\[
\int_{V} \left( \nabla \cdot \mathbf{B} \right) \, dV = 0 \\
\int_{S} \mathbf{B} \cdot d\mathbf{S} = 0 \quad (36)
\]

The magnetic field going into a closed volume equals the magnetic field going out of it.
If the magnetic field is initially divergence free, then it will remain divergence free because of Faraday’s law

Take the divergence of Faraday’s law:

\[
\frac{\partial B}{\partial t} = -c \nabla \times E
\]

\[
\nabla \cdot \left( \frac{\partial B}{\partial t} \right) = \nabla \cdot (-c \nabla \times E)
\]

\[
\frac{\partial}{\partial t} (\nabla \cdot B) = 0
\]

since the divergence of a curl is identically zero. Well, I guess that’s kind of cool.
Writing $\mathbf{B}$ in terms of a vector potential $\mathbf{A}$ automatically satisfies the divergence constraint

- The magnetic field can be written as

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (37)$$

Take the divergence:

$$\nabla \cdot \mathbf{B} = \nabla \cdot \nabla \times \mathbf{A} = 0 \quad (38)$$

- The vector potential formulation allows gauge freedom since $\nabla \times \nabla \phi = 0$ for a scalar function $\phi$. Let $\mathbf{A}' = \mathbf{A} + \nabla \phi$:

$$\mathbf{B} = \nabla \times \mathbf{A}'$$

$$= \nabla \times \mathbf{A} + \nabla \times \nabla \phi$$

$$= \nabla \times \mathbf{A}$$
MHD couples Maxwell’s equations with hydrodynamics to describe macroscopic behavior in highly conducting plasmas.

MHD uses the *low-frequency, long wavelength* approximation.

Each term in the ideal MHD equations has an important physical meaning.

Extensions to MHD are often needed to describe plasma dynamics.

Next up:

- Conservation laws
- Virial theorem
- Extensions to MHD
- Waves, shocks, & instabilities
- More space wombats
Useful References on MHD

- *The Physics of Plasmas* by Boyd & Sanderson (Ch. 3)
- *Magnetohydrodynamics of the Sun* by Priest (Ch. 2)
- *Lectures in Magnetohydrodynamics* by Schnack
  - Lecture 2 reviews the math of plasma physics
  - Lectures 3–9 discuss the equations of MHD
- *Plasma Physics for Astrophysics* by Kulsrud (Ch. 3)