Reduced MHD

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These lecture notes are largely based on Lectures in Magnetohydrodynamics by Dalton Schnack and Nonlinear Magnetohydrodynamics by Dieter Biskamp.
There are many situations where the magnetic field is almost uniform and unidirectional: $\mathbf{B} \approx B_z \mathbf{\hat{z}}$

- Loops in the solar atmosphere
- Magnetic clouds/flux ropes in the solar wind
- Turbulence and transport in strongly magnetized plasmas
- Tokamaks and other magnetically confined fusion devices

Variations along $\mathbf{B}$ are rapidly smoothed out by parallel dynamics

The field is almost potential

The reduced MHD approximation offers a useful simplification so we do not need to solve the full MHD equations
Reduced MHD is applicable when the magnetic field is almost uniform

- Note the vertical scale is smooshed!
Strategy for deriving reduced MHD

▶ Define the magnetic field to be

$$\mathbf{B} = \mathbf{B}_\perp + B_z \hat{z}$$  \hspace{1cm} (1)

▶ Define the small parameter $\varepsilon$ to be

$$\frac{B_\perp}{B_z} \sim \varepsilon \ll 1$$ \hspace{1cm} (2)

▶ Each term in MHD will be ordered as some power of $\varepsilon$

▶ Keep only the lowest powers of $\varepsilon$

▶ Write equations in terms of the flux function and stream function/vorticity

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¹For a full derivation, see §2.5 of *Nonlinear Magnetohydrodynamics* by Biskamp or §13 of *Lectures in Magnetohydrodynamics* by Schnack
The reduced MHD ordering is

\[ V_z = 0, \]

\[ \nabla_\perp \sim 1, \quad \frac{\partial}{\partial z} \sim \varepsilon, \]

\[ D_\eta \sim \varepsilon, \quad V_\perp \sim \varepsilon, \]

\[ p \sim \varepsilon^2, \quad \tilde{B}_z \sim \varepsilon^2. \]  

Example: assume perpendicular dynamics are in approximate energy equipartition

\[ \frac{\rho V_\perp^2}{2} \sim \frac{p}{\gamma - 1} \sim \frac{B_\perp^2}{8\pi} \]  

Since \( B_\perp \sim \varepsilon \) we then have

\[ V_\perp \sim \varepsilon \]  

\[ p \sim \varepsilon^2 \]  

Because \( p \sim \varepsilon^2 \), we ignore pressure dynamics
Assume approximate force balance in parallel direction so that we can choose 

\[ V_z = 0 \] (7)

Most of the spatial structure is in the plane perpendicular to the field so we use the orderings

\[ \nabla_\perp \sim 1, \quad \frac{\partial}{\partial z} \sim \varepsilon \] (8)

The resistivity is small, but not too small:

\[ D_\eta \sim \varepsilon \] (9)

Variations of \( B_z \) are of order \( \varepsilon^2 \) so we ignore them

Assume incompressibility
Define $\mathbf{B}_\perp$ in terms of a flux function, $\psi$

- The magnetic field is given by

$$\mathbf{B} \equiv \mathbf{\hat{z}} \times \nabla \psi + B_{z0}\mathbf{\hat{z}} \quad (10)$$

Here, $\psi = -A_z$ and we choose $A_\perp = 0$

- The divergence constraint is automatically satisfied

- The current density is given by

$$\frac{4\pi}{c} J_z = \nabla^2 \psi \quad (11)$$
Introduce a stream function, $\phi$, and vorticity, $\omega$

- The stream function $\phi$ describes the in-plane flow

\[
\mathbf{V}_\perp = \hat{z} \times \nabla \phi
\]  

(12)

- The in-plane flow is parallel to contours of constant $\phi$
- The in-plane flow is incompressible:

\[
\nabla \cdot \mathbf{V}_\perp = \nabla \cdot (\hat{z} \times \nabla \phi) \\
= \nabla \cdot (\nabla \times \hat{z}) - \hat{z} \cdot (\nabla \times \nabla \phi) \\
= 0
\]  

(13)

so we can choose a constant density, $\rho_0$ (often set to 1)

- Define the vorticity

\[
\omega \equiv \nabla \times \mathbf{V}
\]  

(14)

The $\hat{z}$ component is given by

\[
\omega_z = \nabla^2 \phi
\]  

(15)
The full equations of reduced MHD

The induction and momentum equations become

\[
\frac{\partial \psi}{\partial t} + \mathbf{V}_\perp \cdot \nabla \psi = D_\eta \nabla^2 \psi - B_{z0} \frac{\partial \phi}{\partial z},
\]

\[
\rho_0 \left( \frac{\partial \omega_z}{\partial t} + \mathbf{V}_\perp \cdot \nabla \omega_z \right) = \mathbf{B} \cdot \nabla (J_z).
\]

The vorticity, current density, and perpendicular velocity are

\[
\omega_z = \nabla^2 \perp \phi
\]

\[
\frac{4\pi}{c} J_z = \nabla^2 \psi
\]

\[
\mathbf{V}_\perp = \mathbf{\hat{z}} \times \nabla \phi
\]

There are six equations for six unknowns: \( \psi, \omega_z, \phi, J_z, \mathbf{V}_\perp \)

The equations for vorticity \( \omega_z \) and \( J_z \) are Poisson-type
Takeaway points for reduced MHD

▶ Vector quantities are reduced to scalar functions: $\psi$ and $\phi$
▶ There are no parallel dynamics since these fast time scales have been ordered out of the problem
▶ Pressure dynamics are not included because they are $O(\varepsilon^2)$
▶ Numerical solutions of reduced MHD are more efficient than solving the full equations of MHD
▶ Reduced MHD sometimes allows analytical progress that is not practical in full MHD
Extended Reduced MHD?

- It is possible to include pressure dynamics and compressibility in reduced MHD
  - Use $p \sim \varepsilon$ rather than $p \sim \varepsilon^2$
  - Useful for describing pressure-driven instabilities
- One can also include the Hall term
  - Useful to derive properties of some instabilities
- But the more terms you keep, the less of a simplification reduced MHD is!
- Numerical solution of full MHD is much more practical now than in the past
- Analytical solutions of full MHD are comparably difficult
- Reduced MHD and its extensions still have important applications
There are multiple orderings in plasma physics to describe various phenomena

- **Hall MHD ordering**
  - Fast flows; relatively high frequencies
  - Hall term retained in Ohm’s law
  - Applicable to some highly non-equilibrium situations

- **MHD ordering**
  - Flows comparable to ion thermal speed; low frequencies
  - Applicable to some situations that are not too far from equilibrium

- **Drift ordering**
  - Slow flows; very low frequencies
  - Particle drift velocities are comparable to flow velocities
  - Applicable to situations very close to equilibrium
Application of reduced MHD: heating of the solar corona

- The solar corona has $T \sim 10^6$ K even though the photosphere has $T \sim 6000$ K
- Only the magnetic field has enough energy to heat the plasma
- The two main proposed mechanisms for coronal heating are
  - Nanoflares: small, numerous reconnection events
  - Wave heating: dissipation of MHD waves
- Reduced MHD can be applied to nanoflare heating models through Parker’s problem
Parker’s conjecture

- Start with a uniform magnetic field between two endplates
  - This represents a stretched out/straightened coronal loop
- Mix up the footpoints (e.g., through flows on the endplates)
- Parker conjectured that it will be inevitable for current singularities to form where $J \rightarrow \infty$ if $\eta = 0$
Parker’s conjecture

- This is a difficult problem **analytically**
  - Parker’s conjecture has not been proved
- This is a difficult problem **numerically**
  - Very difficult to simulate infinitely thin structures
- This is a difficult problem **observationally**
  - Magnetic fields very difficult to observe in the corona
  - Energy dissipation and heating occurs on tiny scales!
Parker’s conjecture

- Where does the energy go?
  - Source is kinetic energy of photospheric motions
  - The kinetic energy is stored as magnetic energy
  - Magnetic energy is dissipated into thermal energy
- Parker predicted quasi-static evolution between force-free states
- Parker’s problem is not purely academic
  - There are real implications for how magnetic energy is dissipated in stellar coronae
  - This problem helps us understand how small-scale magnetic reconnection occurs in coronal loops
Reduced MHD simulations show the formation of current filaments (Ng et al. 2012)

- Flows on the endplates twist up the field
- Resistive dissipation releases stored magnetic energy
How does heating depend on Lundquist number?

- As $\eta \to 0$, the heating rate starts to plateau.
- Does the heating rate become independent of Lundquist number as $S \to \infty$?
- Resistive heating is given by

$$Q_{tot} = \int_{V} \eta J^2 \, dV$$

(21)

As $\eta$ becomes smaller, the peak current increases and encompassing volume decreases.
Taylor relaxation

- Taylor hypothesized that relaxation occurs by the magnetic field finding its minimum energy state while conserving helicity.
- The final state will be force-free with constant $\alpha$:
  - A linear force-free field!
- In general, it is not possible to relax to a linear force-free field without changing the magnetic topology:
  - Resistive diffusion or reconnection is required.
  - This dissipation heats the plasma.
Reduced MHD is applicable in situations where the magnetic field is almost uniform
  ▶ Coronal loops
  ▶ Tokamaks

Reduced MHD puts the equations in terms of a flux function $\psi$ and a stream function $\phi$ (with associated vorticity $\omega_z$)

Reduced MHD reduces computational requirements and simplifies analytical theory

Investigations of Parker’s conjecture provide insight into the nanoflare mechanism of coronal heating