# Reduced MHD

Nick Murphy

Harvard-Smithsonian Center for Astrophysics

Astronomy 253: Plasma Astrophysics

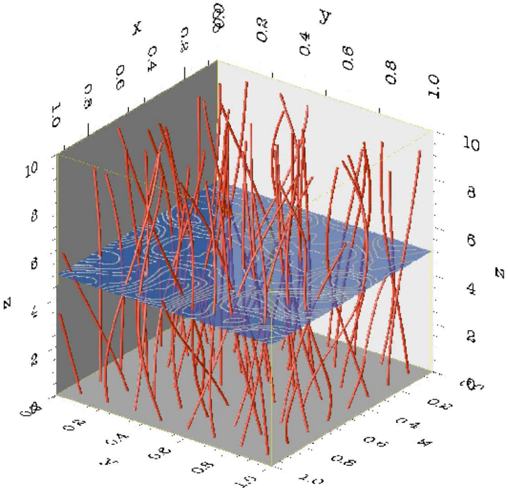
February 10, 2016

These lecture notes are largely based on *Lectures in Magnetohydrodynamics* by Dalton Schnack and *Nonlinear Magnetohydrodynamics* by Dieter Biskamp.

# Reduced MHD

- There are many situations where the magnetic field is almost uniform and unidirectional: $\mathbf{B} \approx B_z \hat{\mathbf{z}}$
  - Loops in the solar atmosphere
  - Magnetic clouds/flux ropes in the solar wind
  - Turbulence and transport in strongly magnetized plasmas
  - Tokamaks and other magnetically confined fusion devices
- Variations along $\mathbf{B}$ are rapidly smoothed out by parallel dynamics
- The field is almost potential
- The reduced MHD approximation offers a useful simplification so we do not need to solve the full MHD equations

# Reduced MHD is applicable when the magnetic field is almost uniform




- Note the vertical scale is smooshed!

# Strategy for deriving reduced MHD[1]

- Define the magnetic field to be

$$\mathbf{B} = \mathbf{B}_\perp + B_z \hat{\mathbf{z}} \tag{1}$$

- Define the small parameter $\varepsilon$ to be

$$\frac{B_\perp}{B_z} \sim \varepsilon \ll 1 \tag{2}$$

- Each term in MHD will be ordered as some power of $\varepsilon$
- Keep only the lowest powers of $\varepsilon$
- Write equations in terms of the flux function and stream function/vorticity

[1] For a full derivation, see §2.5 of *Nonlinear Magnetohydrodynamics* by Biskamp or §13 of *Lectures in Magnetohydrodynamics* by Schnack

# Orderings of reduced MHD

- The reduced MHD ordering is

$$\begin{aligned} &V_z = 0, \\ \nabla_\perp \sim 1, \quad &\frac{\partial}{\partial z} \sim \varepsilon, \\ D_\eta \sim \varepsilon, \quad &V_\perp \sim \varepsilon, \\ p \sim \varepsilon^2, \quad &\tilde{B}_z \sim \varepsilon^2. \end{aligned} \tag{3}$$

- Example: assume perpendicular dynamics are in approximate energy equipartition

$$\frac{\rho V_\perp^2}{2} \sim \frac{p}{\gamma - 1} \sim \frac{B_\perp^2}{8\pi} \tag{4}$$

Since $B_\perp \sim \varepsilon$ we then have

$$V_\perp \sim \varepsilon \tag{5}$$

$$p \sim \varepsilon^2 \tag{6}$$

Because $p \sim \varepsilon^2$, we ignore pressure dynamics

# Orderings of reduced MHD

- Assume approximate force balance in parallel direction so that we can choose
$$V_z = 0 \tag{7}$$
- Most of the spatial structure is in the plane perpendicular to the field so we use the orderings
$$\nabla_\perp \sim 1, \frac{\partial}{\partial z} \sim \varepsilon \tag{8}$$
- The resistivity is small, but not too small:
$$D_\eta \sim \varepsilon \tag{9}$$
- Variations of $B_z$ are of order $\varepsilon^2$ so we ignore them
- Assume incompressibility

# Define $\mathbf{B}_\perp$ in terms of a flux function, $\psi$

- The magnetic field is given by

$$\mathbf{B} \equiv \hat{\mathbf{z}} \times \nabla\psi + B_{z0}\hat{\mathbf{z}} \tag{10}$$

  Here, $\psi = -A_z$ and we choose $\mathbf{A}_\perp = 0$
- The divergence constraint is automatically satisfied
- The current density is given by

$$\frac{4\pi}{c} J_z = \nabla^2\psi \tag{11}$$

# Introduce a stream function, $\phi$, and vorticity, $\boldsymbol{\omega}$

- The stream function $\phi$ describes the in-plane flow

$$\mathbf{V}_\perp = \hat{\mathbf{z}} \times \nabla\phi \tag{12}$$

- The in-plane flow is parallel to contours of constant $\phi$
- The in-plane flow is incompressible:

$$\begin{aligned} \nabla \cdot \mathbf{V}_\perp &= \nabla \cdot (\hat{\mathbf{z}} \times \nabla\phi) \\ &= \nabla \cdot (\nabla \times \hat{\mathbf{z}}) - \hat{\mathbf{z}} \cdot (\nabla \times \nabla\phi) \\ &= 0 \end{aligned} \tag{13}$$

  so we can choose a constant density, $\rho_0$ (often set to 1)
- Define the vorticity

$$\boldsymbol{\omega} \equiv \nabla \times \mathbf{V} \tag{14}$$

  The $\hat{\mathbf{z}}$ component is given by

$$\omega_z = \nabla_\perp^2 \phi \tag{15}$$

# The full equations of reduced MHD

- The induction and momentum equations become

$$\frac{\partial \psi}{\partial t} + \mathbf{V}_\perp \cdot \nabla \psi = D_\eta \nabla^2 \psi - B_{z0} \frac{\partial \phi}{\partial z}, \tag{16}$$

$$\rho_0 \left( \frac{\partial \omega_z}{\partial t} + \mathbf{V}_\perp \cdot \nabla \omega_z \right) = \mathbf{B} \cdot \nabla \left( J_z \right). \tag{17}$$

The vorticity, current density, and perpendicular velocity are

$$\omega_z = \nabla_\perp^2 \phi \tag{18}$$

$$\frac{4\pi}{c} J_z = \nabla^2 \psi \tag{19}$$

$$\mathbf{V}_\perp = \hat{\mathbf{z}} \times \nabla \phi \tag{20}$$

- There are six equations for six unknowns: $\psi$, $\omega_z$, $\phi$, $J_z$, $\mathbf{V}_\perp$
- The equations for vorticity $\omega_z$ and $J_z$ are Poisson-type

# Takeaway points for reduced MHD

- Vector quantities are reduced to scalar functions: $\psi$ and $\phi$
- There are no parallel dynamics since these fast time scales have been ordered out of the problem
- Pressure dynamics are not included because they are $\mathcal{O}(\varepsilon^2)$
- Numerical solutions of reduced MHD are more efficient than solving the full equations of MHD
- Reduced MHD sometimes allows analytical progress that is not practical in full MHD

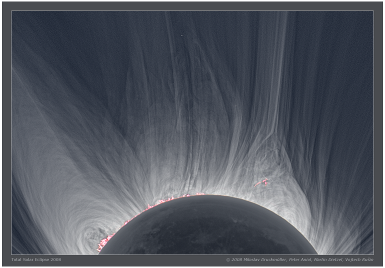
# Extended Reduced MHD?

- It is possible to include pressure dynamics and compressibility in reduced MHD
  - Use $p \sim \varepsilon$ rather than $p \sim \varepsilon^2$
  - Useful for describing pressure-driven instabilities
- One can also include the Hall term
  - Useful to derive properties of some instabilities
- But the more terms you keep, the less of a simplification reduced MHD is!
- Numerical solution of full MHD is much more practical now than in the past
- Analytical solutions of full MHD are comparably difficult
- Reduced MHD and its extensions still have important applications

# There are multiple orderings in plasma physics to describe various phenomena

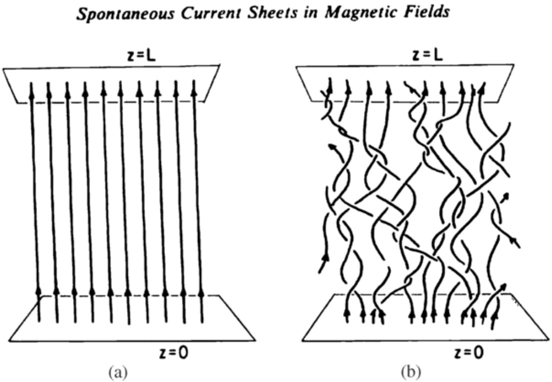
- Hall MHD ordering
  - Fast flows; relatively high frequencies
  - Hall term retained in Ohm's law
  - Applicable to some highly non-equilibrium situations
- MHD ordering
  - Flows comparable to ion thermal speed; low frequencies
  - Applicable to some situations that are not too far from equilibrium
- Drift ordering
  - Slow flows; very low frequencies
  - Particle drift velocities are comparable to flow velocities
  - Applicable to situations very close to equilibrium

# Application of reduced MHD: heating of the solar corona




- The solar corona has $T \sim 10^6$ K even though the photosphere has $T \sim 6000$ K
- Only the magnetic field has enough energy to heat the plasma
- The two main proposed mechanisms for coronal heating are
  - Nanoflares: small, numerous reconnection events
  - Wave heating: dissipation of MHD waves
- Reduced MHD can be applied to nanoflare heating models through Parker's problem

# Parker's conjecture




- Start with a uniform magnetic field between two endplates
  - This represents a stretched out/straightened coronal loop
- Mix up the footpoints (e.g., through flows on the endplates)
- Parker conjectured that it will be inevitable for current singularities to form where $\mathbf{J} \to \infty$ if $\eta = 0$

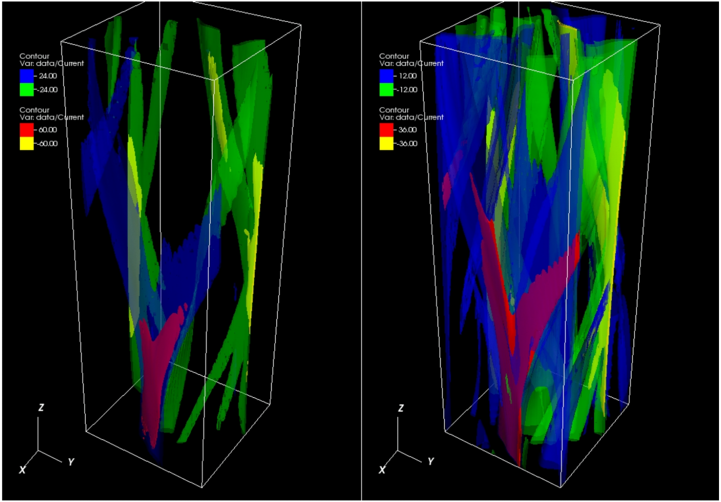
# Parker's conjecture

- This is a difficult problem **analytically**
  - Parker's conjecture has not been proved
- This is a difficult problem **numerically**
  - Very difficult to simulate infinitely thin structures
- This is a difficult problem **observationally**
  - Magnetic fields very difficult to observe in the corona
  - Energy dissipation and heating occurs on tiny scales!

# Parker's conjecture

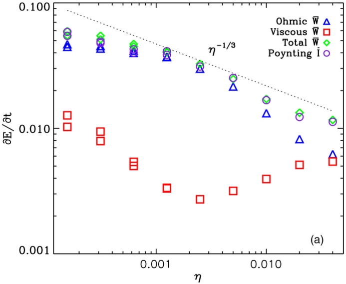
- Where does the energy go?
  - Source is kinetic energy of photospheric motions
  - The kinetic energy is stored as magnetic energy
  - Magnetic energy is dissipated into thermal energy
- Parker predicted quasi-static evolution between force-free states
- Parker's problem is not purely academic
  - There are real implications for how magnetic energy is dissipated in stellar coronae
  - This problem helps us understand how small-scale magnetic reconnection occurs in coronal loops

# Reduced MHD simulations show the formation of current filaments (Ng et al. 2012)




- Flows on the endplates twist up the field
- Resistive dissipation releases stored magnetic energy

# How does heating depend on Lundquist number?




- As $\eta \to 0$, the heating rate starts to plateau
- Does the heating rate become independent of Lundquist number as $S \to \infty$?
- Resistive heating is given by

$$Q_{tot} = \int_{\mathcal{V}} \eta J^2 \, \mathrm{d}\mathcal{V} \tag{21}$$

As $\eta$ becomes smaller, the peak current increases and encompassing volume decreases

# Taylor relaxation

- Taylor hypothesized that relaxation occurs by the magnetic field finding its minimum energy state while conserving helicity
- The final state will be force-free with constant $\alpha$
  - A linear force-free field!
- In general, it is not possible to relax to a linear force-free field without changing the magnetic topology
  - Resistive diffusion or reconnection is required
  - This dissipation heats the plasma

# Summary

- Reduced MHD is applicable in situations where the magnetic field is almost uniform
  - Coronal loops
  - Tokamaks
- Reduced MHD puts the equations in terms of a flux function $\psi$ and a stream function $\phi$ (with associated vorticity $\omega_z$)
- Reduced MHD reduces computational requirements and simplifies analytical theory
- Investigations of Parker's conjecture provide insight into the nanoflare mechanism of coronal heating