Magnetic Reconnection with Asymmetry in the Outflow Direction

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50th Annual Meeting of the APS Division of Plasma Physics
November 17-21, 2008
Dallas, Texas
Outline

Motivation for studying reconnection with asymmetry in the outflow direction

Presentation of the equations of steady-state resistive MHD in surface integral form

Review of Sweet-Parker reconnection when pressure effects and compressibility are included

Control volume analysis of reconnection with asymmetric downstream pressure

Control volume analysis of reconnection in toroidal geometry with the outflow aligned with the radial direction

Numerical tests of this model in straight and toroidal geometries
Reconnection with asymmetry in the outflow direction

- Reconnection in physically realistic scenarios will often have asymmetry in the outflow direction
  - Current sheets in the near-Earth magnetotail
  - Coronal mass ejections and some solar flares
  - Laboratory merging of toroidal plasma configurations (e.g., SSX, MRX, TS-3/4/5)
  - Magnetically channeled disks in the winds of massive stars
  - Reconnection during turbulence or reconnection with multiple competing islands

- Cassak & Shay (2007) extend the Sweet-Parker model to describe reconnection between plasmas with different upstream magnetic field strengths and/or different densities

- We perform a similar analysis for reconnection with asymmetric downstream pressure
Resistive MHD model

The equations of resistive MHD in conservative form are

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0 \]  
\[ \frac{\partial (\rho \mathbf{V})}{\partial t} + \nabla \cdot \left[ \rho \mathbf{VV} + \left( p + \frac{B^2}{2\mu_0} \right) \mathbf{i} - \frac{\mathbf{BB}}{\mu_0} \right] = 0 \]  
\[ \frac{\partial e}{\partial t} + \nabla \cdot \left[ \left( \frac{\rho V^2}{2} + \frac{\gamma}{\gamma - 1} p \right) \mathbf{V} + \frac{\mathbf{E} \times \mathbf{B}}{\mu_0} \right] = 0 \]  
\[ \frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0 \]  
\[ \mu_0 \mathbf{J} = \nabla \times \mathbf{B} \]  
\[ \mathbf{E} + \mathbf{V} \times \mathbf{B} = \eta \mathbf{J} \]

In order, these represent conservation of mass, momentum, and energy, Faraday’s law, Ampere’s law without displacement current, and Ohm’s law, with \( e \equiv \rho V^2/2 + p/(\gamma - 1) + B^2/2\mu_0 \).
Equations of steady-state MHD in integral form

With the help of Gauss’ and Stokes’ theorems, the equations of steady-state resistive MHD can be written as

\( \oint_S dS \cdot (\rho \mathbf{V}) = 0 \) (7)

\( \oint_S dS \cdot \left[ \rho \mathbf{V} \cdot \mathbf{V} + \left( p + \frac{B^2}{2\mu_0} \right) \hat{\mathbf{I}} - \frac{\mathbf{B} \times \mathbf{B}}{\mu_0} \right] = 0 \) (8)

\( \oint_S dS \cdot \left[ \left( \frac{\rho V^2}{2} + \frac{\gamma p}{\gamma - 1} \right) \mathbf{V} + \left( \frac{\mathbf{E} \times \mathbf{B}}{\mu_0} \right) \right] = 0 \) (9)

\( \oint_S dS \times \mathbf{E} = 0 \) (10)

In order, these represent conservation of mass, momentum, energy, and flux (e.g., Goedbloed & Poedts 2004)

These relations are valid for any closed volume in steady-state MHD.
Symmetric downstream pressure

The effects of symmetric downstream pressure have been analyzed in Chapter 4 of Priest & Forbes (2000). We repeat this analysis without assuming incompressibility.

Conservation of mass gives

\[ \rho_{\text{in}} V_{\text{in}} L \sim \rho_{\text{out}} V_{\text{out}} \delta \quad (11) \]

Defining \( \alpha \equiv \gamma / (\gamma - 1) \) and using \( E_y = V_{\text{in}} B_{\text{in}} \), conservation of energy gives

\[ V_{\text{in}} L \left( \alpha p_{\text{in}} + \frac{B_{\text{in}}^2}{\mu_0} \right) \sim V_{\text{out}} \delta \left( \frac{\rho_{\text{out}} V_{\text{out}}^2}{2} + \alpha p_{\text{out}} \right) \quad (12) \]

Defining \( V_A \equiv B_{\text{in}} / \sqrt{\mu_0 \rho_{\text{in}}} \), the outflow velocity is then given by

\[ V_{\text{out}}^2 \sim 2V_A^2 - 2\alpha \left( \frac{p_{\text{out}}}{\rho_{\text{out}}} - \frac{p_{\text{in}}}{\rho_{\text{in}}} \right) \quad (13) \]
Symmetric downstream pressure (cont.)

- The outflow will be given by $V_{out} \sim V_A$ when

$$\alpha \left[ p_{out} \left( \frac{\rho_{in}}{\rho_{out}} \right) - p_{in} \right] \sim \frac{B_{in}^2}{2\mu_0} \tag{14}$$

- The results so far are independent of dissipation mechanism, but to derive a reconnection rate, we now assume resistive dissipation.

- Because $E_y = V_{in}B_{in}$ is constant and $J_y \sim B_{in}/\delta$, the inflow velocity is given by

$$V_{in} \sim \frac{\eta}{\mu_0 \delta} \tag{15}$$

- Using conservation of mass and that $S \equiv \mu_0 LV_A/\eta$, we see that

$$\frac{V_{in}}{V_A} \sim \frac{\rho_{out}}{\rho_{in}} \frac{\delta}{L} \frac{V_{out}}{V_A} \Rightarrow \frac{V_{in}}{V_A} \sim \sqrt{\frac{\rho_{out} V_{out} 1}{\rho_{in} V_A S}} \tag{16}$$
Symmetric downstream pressure (cont.)

The reconnection rate for compressible Sweet-Parker-like reconnection is then given by

\[
\frac{V_{in}}{V_A} \sim \frac{2^{1/4}}{S^{1/2}} \sqrt{\frac{\rho_{out}}{\rho_{in}}} \left[ 1 - \frac{\alpha}{2} \left( \frac{p_{out} (\rho_{in}/\rho_{out}) - p_{in}}{B_{in}^2/2\mu_0} \right) \right]^{1/4}
\]

In this analysis, we include the full Poynting flux rather than just the inflowing magnetic energy.

Reconnection will not be able to occur when

\[
\frac{B_{in}^2}{2\mu_0} \lesssim \frac{\alpha}{2} \left[ p_{out} \left( \frac{\rho_{in}}{\rho_{out}} \right) - p_{in} \right]
\]

Compressibility effects can reduce the bottleneck caused by conservation of mass.

The reconnection rate is weakly sensitive to downstream pressure because the current sheet width $\delta$ is allowed to increase.
Modifying the S-P analysis for asymmetric outflow

The above figure represents a current sheet with asymmetric downstream pressure \( p_L > p_R \).

The current sheet length is given by \( 2\mathcal{L} \equiv L_L + \epsilon + L_R \).

The solid vertical bar represents the flow stagnation point, and the dashed bar represents the magnetic field null.

We assume the reconnection process is externally driven in such a way that the current sheet position remains static.
Finding scaling relations for asymmetric outflow

We evaluate the surface integrals given in Eqs. 7–10 over the entire current sheet.

Conservation of mass gives

\[ 2\rho_{in} V_{in} \mathcal{L} \sim \rho L V_L \delta + \rho_R V_R \delta \] (19)

Conservation of momentum in the outflow direction gives

\[ \rho L V_L^2 + p_L \sim \rho R V_R^2 + p_R \] (20)

Ignoring upstream kinetic energy and downstream magnetic energy, conservation of energy gives

\[ 2V_{in} \mathcal{L} \left( \alpha p_{in} + \frac{B_{in}^2}{\mu_0} \right) \sim V_L \delta \left( \alpha p_L + \frac{\rho L V_L^2}{2} \right) + V_R \delta \left( \alpha p_R + \frac{\rho R V_R^2}{2} \right) \] (21)
Finding a cubic relationship for $V_L^2$ as a function of $p_L$ and $p_R$

By using conservation of mass to eliminate $2L V_{in}$ and conservation of momentum to eliminate $V_R$ in the relationship for conservation of energy, we arrive at the cubic relationship in $V_L^2$

\begin{equation}
0 \sim C_{6L} V_L^6 + C_{4L} V_L^4 + C_{2L} V_L^2 + C_{0L}
\end{equation}

The expressions for the coefficients are given on the next slide.

This can be solved analytically using Cardano’s method, or numerically by root-solving.

The above relation depends on downstream parameters including density and pressure.

The above expression does not depend on dissipation mechanism, but it does assume a single-fluid framework.
Coefficients for the cubic relationship for $V_L^2$

(23) $C_{0L} \equiv \frac{\rho_R}{\rho_{in}^2} \left( \alpha p_{in} + \frac{B_{in}^2}{\mu_0} \right)^2 (p_L - p_R)$

\[ + \left( \alpha p_{in} + \frac{B_{in}^2}{\mu_0} \right) \left( \frac{p_R - p_L}{\rho_{in}} \right) (p_L + 2\alpha p_R - p_R) \]

\[ + \left( \frac{p_L - p_R}{4\rho_R} \right) (p_L + 2\alpha p_R - p_R)^2 \]

(24) $C_{2L} \equiv \left( \alpha p_{in} + \frac{B_{in}^2}{\mu_0} \right)^2 \frac{\rho_L}{\rho_{in}^2} \left( \frac{\rho_R - \rho_L}{\rho_{in}} \right)$

\[ + \left( \alpha p_{in} + \frac{B_{in}^2}{\mu_0} \right) \frac{2\rho_L (p_R - p_L)(1 - \alpha)}{\rho_{in}} \]

\[ + \frac{\rho_L}{4\rho_R} (p_L + 2\alpha p_R - p_R)(3p_L - 3p_R + 2\alpha p_R) - \alpha^2 p_L^2 \]

(25) $C_{4L} \equiv \frac{\rho_{in}^2}{\rho_R} \left[ \frac{3}{4} (p_L - p_R) + \alpha p_R \right] - \alpha p_L \rho_L$

(26) $C_{6L} \equiv \frac{1}{4} \left( \frac{\rho_L^3}{\rho_R} - \rho_{in}^2 \right)$
A simpler result is found by assuming incompressibility

By assuming incompressibility \((\rho_L = \rho_R = \rho_{in} \equiv \rho, \alpha \to 1)\), the cubic polynomial in \(V_L^2\) reduces to

\[
0 \sim V_L^4 + C_2 V_L^2 + C_0.
\]

(27)

Here, the coefficient \(C_{2L}\) is given by

\[
C_{2L} \equiv \frac{p_L - p_R}{\rho},
\]

(28)

and the coefficient \(C_{0L}\) is given by

\[
C_{0L} \equiv 4p_{in} \left( \frac{p_L + p_R}{\rho^2} \right) - \left( \frac{2p_{in}}{\rho} \right)^2 - \left( \frac{p_L + p_R}{\rho} \right)^2
\]

\[
+ 4V_A^2 \left( \frac{p_L + p_R - 2p_{in}}{\rho} \right) - 4V_A^4.
\]

(29)

This result is less accurate, but more illustrative and with fewer free parameters.
Deriving the outflow velocities (incompressible case)

- An expression for $V_L^2$ is easily found,

\begin{equation}
V_L^2 \sim -\frac{C_{2L}}{2} + \frac{\sqrt{C_{2L}^2 - 4C_{0L}}}{2}.
\end{equation}

- Then, using conservation of momentum, $V_R^2$ is given by

\begin{equation}
V_R^2 \sim V_L^2 + \left(\frac{p_L - p_R}{\rho}\right).
\end{equation}

- By assuming resistive dissipation, the reconnection rate can be expressed as

\begin{equation}
\frac{V_{\text{in}}}{V_A} = \sqrt{\frac{V_L + V_R}{2V_A S}}
\end{equation}

after $V_L$ and $V_R$ have been found.
Solution plots for $|V_L|$ and $S^{1/2}V_{in}/V_A$ (incompressible case)

**Left:** Solution contours for the magnitude of the leftward-directed outflow velocity $|V_L|$ as a function of $p_L$ and $p_R$ with $p_{in} = 0$. Contours are separated by $0.25V_A$. The dashed line represents when $V_L = 0$.

**Right:** Solution contours for the normalized reconnection rate $S^{1/2}V_{in}/V_A$. Contours are separated by $0.25$. The dashed line represents when $S^{1/2}V_{in}/V_A = 0$.

- The reconnection rate is greatly affected only when outflow from both sides of the current sheet is blocked. The current sheet width $\delta$ is longer for greater downstream pressure.
The flow stagnation point is found through conservation of mass

Conservation of mass inside the current sheet gives the relations

\[(33) \quad \rho_{in} V_{in} L_L \sim \rho_L V_L \delta,\]
\[(34) \quad \rho_{in} V_{in} (\epsilon + L_R) \sim \rho_R V_R \delta,\]

where \(V_n\) is the flow velocity at the magnetic field null.

The position of the flow stagnation point is then given by

\[(35) \quad L_L \sim 2\mathcal{L} \left( \frac{\rho_L V_L}{\rho_L V_L + \rho_R V_R} \right),\]
\[(36) \quad \epsilon + L_R \sim 2\mathcal{L} \left( \frac{\rho_R V_R}{\rho_L V_L + \rho_R V_R} \right).\]
The magnetic field null is approximated with the momentum equation

The outflow component of the momentum equation (ignoring magnetic pressure) is given by

\[ \rho V_x \frac{\partial V_x}{\partial x} = \frac{B_z}{\mu_0} \frac{\partial B_x}{\partial z} - \frac{\partial p}{\partial x} \]  

(37)

Evaluating this at the flow stagnation point, \( x_s \), and using \( \frac{\partial B_x}{\partial z} \sim \frac{B_{in}}{\delta} \), we find

\[ \frac{B_z(x_s)}{\mu_0} \frac{B_{in}}{\delta} \sim \frac{\partial p}{\partial x} \bigg|_{x=x_s} \]

(38)

Given that \( B_z(x_n) = 0 \) at the magnetic field null position, \( x_n \), a Taylor expansion around \( x = x_n \) gives

\[ B_z(x_s) = (x_n - x_s) \frac{\partial B_z}{\partial x} \bigg|_{x_n} + \mathcal{O}[(x_n - x_s)^2] \]

We then approximate the position of the magnetic field null by

\[ \epsilon \equiv x_n - x_s \sim -\frac{\mu_0 \delta}{B_{in}} \left( \frac{\partial p/\partial x}{\partial B_z/\partial x} \right)_{x=x_s} \]

(39)

The field null and stagnation point will not coincide unless the pressure gradient is zero at the flow stagnation point in a steady-state.

When the two points are separated, there will be a Poynting flux across the flow stagnation point even though \( V = 0 \) there.
Toroidal geometry: setup

We also consider reconnection in a toroidal geometry where the outflow is aligned with the radial direction.

- Inflow occurs on two annuli defined by $R \in [R_1, R_2]$ for $Z = \pm \delta$
- Outflow occurs on two cylinders defined by $Z \in [-\delta, \delta]$ for $R \in \{R_1, R_2\}$

From the divergence constraint, we assume that the upstream magnetic field takes the form $B_{in}(R) = B_0 R_0 / R$, where $B_0$ is the magnetic field strength at radius $R_0$. 
Toroidal geometry: scaling relations

- By integrating over the entire volume, conservation of mass gives

\[
\rho_{in} V_{in} \left( \pi R_2^2 - \pi R_1^2 \right) \sim 2\pi R_1 \delta \rho_1 V_1 + 2\pi R_2 \delta \rho_2 V_2
\]  
(40)

- Conservation of momentum gives

\[
R_1 \left( \rho_1 V_1^2 + p_1 \right) \sim R_2 \left( \rho_2 V_2^2 + p_2 \right)
\]
(41)

- Using \( B_{in}(R) = B_0 R_0 / R \), conservation of energy gives

\[
(\pi R_2^2 - \pi R_1^2) V_{in} \alpha p_{in} + \frac{2\pi V_{in} B_0^2 R_2^2}{\mu_0} \ln \left( \frac{R_2}{R_1} \right) \sim \]

\[
2\pi R_1 \delta V_1 \left( \frac{\rho_1 V_1^2}{2} + \alpha p_1 \right) + 2\pi R_2 \delta V_2 \left( \frac{\rho_2 V_2^2}{2} + \alpha p_2 \right)
\]
(42)

- The inflow velocity and current sheet width remain related by

\[
V_{in} \sim \eta / \mu_0 \delta
\]
Effects associated with toroidal geometry are applicable to low aspect ratio spheromak merging (SSX, MRX, TS-3/4/5, etc.)

The relation for momentum balance shows that it is possible to have quicker inward-directed outflow than outward-directed outflow even if the inboard pressure exceeds the outboard pressure.

This effect has been seen in simulations of MRX (Murphy & Sovinec 2008)

In these situations, it is not always reasonable to assume that the outflow from both sides of the current sheet will be the same.

Conservation of mass inside the current sheet gives the flow stagnation radius

\[
R_s \sim \sqrt{\frac{\rho_1 V_1 R_1 R_2^2 - \rho_2 V_2 R_2 R_1^2}{\rho_1 V_1 R_1 + \rho_2 V_2 R_2}}
\]

(43)
Testing the model through simulations

In order to test these models, we perform resistive MHD simulations using the NIMROD finite element code (Sovinec et al. 2004).

The first strategy is to initialize a singly periodic Harris sheet equilibrium with two perturbations closer to each other in one direction than the other.

- The current sheet position is free to move after the current sheets start interacting.

The second strategy is to perform linear geometry simulations using the grid established previously of MRX (Murphy & Sovinec 2008), except with one downstream wall closer than the other downstream wall.

- The driving mechanism constrains the current sheet position.

To test toroidal geometry simulations, we compare the theory developed in this poster with the simulations of MRX in toroidal geometry.
Double perturbation simulations of a Harris sheet show X-line retreat

The position of the current sheet drifts away from the central pressure buildup at $\sim 0.1V_A$, consistent with kinetic simulations performed by Oka et al. in a forthcoming PRL.

Simulation parameters are $B_0 = 250$ G, $V_A \approx 77$ km/s, $\delta_0 = 1$ cm, $B_x = B_0 \tanh (z/\delta_0)$, $S \sim 1000$, $m_i = m_p$, $L \sim 1$ m, $\beta_{in} = 2$, and $n = 5 \times 10^{19}$ m$^{-3}$.
Double perturbation simulations

From left to right are pressure, outflow velocity, and out of plane current density for a singly periodic Harris sheet simulation with two initial perturbations close to each other.
Conservation of momentum shows good agreement.

Conservation of mass and energy show mediocre agreement, likely due to time-dependent behavior such as X-line retreat and lengthening of the current sheet.

Values were chosen from the simulation at where the current density goes down by a factor of $e$ from its peak value. These points represent different times during the same simulation.

A voltage application to drive reconnection should help constrain the current sheet’s position.
We examine simulations of MRX in multiple geometries as a test

Simulations of the Magnetic Reconnection Experiment (MRX) were performed in Murphy & Sovinec (2008)

During the push mode of operation (similar to spheromak merging), a density buildup at low radii occurred because less volume was available on the inboard side of the current sheet than the outboard side, pushing the X-point towards higher radii and contributing to an asymmetric outflow pattern

The position of the current sheet does not greatly vary because reconnection was driven through flux cores

In linear geometry, the outer boundary is rectangular
Comparison of linear geometry MRX simulation to Eqs. 19–21

We present comparisons of the conservation relations during a simulation of push reconnection in linear geometry using the setup of MRX where one wall is closer to the current sheet than the other wall.

Here and in the next slide, we calculate $\delta$ and $L$ and choose values from when the current density falls by a factor of $e$ from its peak value.

The data points represent different times during the same simulation.

Agreement with Eqs. 19–21 is good, especially considering the tension force in the downstream region is non-negligible.
Comparison of toroidal geometry MRX simulation to Eqs. 40–42

We see fair agreement with conservation of mass and energy at different times during the push reconnection simulation reported in Murphy & Sovinec (2008)

The discrepancy regarding conservation of mass is because the width of the current sheet increases further from the X-point

The discrepancy regarding conservation of energy is because our assumed magnetic field profile overestimates the Poynting flux into the layer
Summary and Conclusions

- Magnetic reconnection with asymmetry in the outflow direction occurs in many situations in nature and the laboratory.
- By using a surface integral approach, we derive scaling relationships that describe reconnection with asymmetric downstream pressure.
- Reconnection is greatly slowed only when outflow from both sides of the current sheet is blocked because current sheet broadening reduces the bottleneck effect associated with conservation of mass.
- A similar analysis is possible for toroidal geometry.
  - In this case, it is possible to have quicker radially inward directed outflow than radially outward directed outflow even when the inboard pressure exceeds the outboard pressure.
- NIMROD simulations show reasonably good agreement with our order of magnitude theory regarding conservation of mass, momentum, and energy.
- Future work involves improving comparisons to simulation and investigating the current sheet interior.