5.4 CONVECTION

Two approaches: (1) hydrodynamical & (2) thermodynamical, and two views: (4) transport & (2) instability - the bias of Victorian scientists, paddy

Prototypical instability - sound waves: \(-\omega^2 + \alpha^2 k^2 = 0\) (real roots)

\(\rightarrow\) add gravity - gravity waves (pressure acts like longitudinal sound wave, pressure works like a restoring force)

For an instability to grow, the frequency should be either purely imaginary or complex, with the imaginary being negative.

Instability arises in systems which have at least two characteristic time scales, e.g. sound travel time and free-fall time - a critical parameter, which often of order unity indicates the onset of, e.g. Jeans instability.

Consider a plane-parallel fluid, under gravity with accel. \(g\).

Buoyancy:

A parcel of fluid (sphere or cube) of volume \(V \sim l^3\) is in pressure equilibrium with the surrounding medium, because the sound traversal time \(\sim l/v\) is short compared to the parcel "survival" time.

If the temperature of the parcel, \(T' > T\) of surroundings, \(\Rightarrow p' < p\), and the parcel will get a net upward buoyancy force: \((\rho - \rho')Vg\)

Stability question -

adiabatic perturbative motions for simplicity to start with.
Oct. 17: Start by drawing

\[ \frac{dT}{dz} \] does not have to be adiabatic

\[ z = z_R \]

\[ \frac{dT}{dz} \] unstable
First calculate \( \frac{dT}{dz} \) without convection; all heat carried radiatively.

The \( T \) gradient:

\[
\frac{dT}{dz} = \frac{dT}{dp} \frac{dp}{dz} = T \frac{d\ln T}{dp} \frac{dp}{dz}
\]

\( P = p' \)

in parcel:

\[
\frac{dT'}{dz} = - \frac{T'}{\lambda_p} \frac{d\ln T'}{dp} \frac{dp}{dz}
\]

\( T' = T \) — small perturbation

Convectionally unstable:

\[
\frac{dT'}{dp} > \frac{d\ln T'}{dp}
\]

i.e. \( T' \) decreases more slowly than \( T \) as parcel rises.

Or in other words, the perturbation that causes the parcel to rise takes place in an environment that encourages further rising.

Alternatively — conv. unst.:

\[
\frac{d\ln T}{dp} > \frac{d\ln T'}{dp}
\]

NB: Note that these are local criteria.

Alternatively — conv. unst.:

\[
\frac{dS}{dz} < 0
\]

we have prepared the stage for

**Mixing Length Theory**

"parcel survive over a characteristic distance, \( \ell \), mixing length \( \equiv \) parcel \( \ell \)

**DERIVE** the equations of motion for a leaky parcel.

(A) Radiative Leakage

Parcel heat content

\[
\frac{dQ'}{dt} = - \nabla \cdot F_{rad}
\]

(Eulerian)
\[
\frac{\partial T'}{\partial t} = -\frac{1}{\rho c_p} \nabla \cdot \mathbf{F}_{\text{rad}}
\]

\[
\frac{\partial T'}{\partial t} = \frac{4a c T^3}{3k\rho^2 c_p} \quad \text{thermal diffusivity} \quad [\text{cm}^2/\text{s}]
\]

\[
\Rightarrow \text{the characteristic radiative cooling time for a parcel is} \quad \frac{\rho}{\nu_t}
\]

\[
\frac{\partial^2 T'}{\partial^2 z} \quad \text{Lagrangian:} \quad \frac{\partial T'}{\partial t} = \frac{\partial T'}{\partial t} + \mathbf{v}(\frac{\partial T'}{\partial z})
\]

\[
\frac{\partial T'}{\partial z} = \frac{d}{dz}(\frac{dT'}{dz})_{\text{ad}}
\]

\[
\Rightarrow \frac{\partial T'}{\partial t} = \frac{v_T}{\rho} (T - T') + \mathbf{v}(\frac{dT'}{dz})_{\text{ad}}
\]

The vertical rate of change of ambient \(T\) as seen by the moving parcel

\[
\frac{\partial T}{\partial t} = \frac{dT}{dz} \quad \text{Eqn. Energy}
\]

\[
\frac{D\Delta T}{Dt} = -\frac{v_T}{\rho} \Delta T + \mathbf{v} \left[ \left( \frac{dT}{dz} \right)_{\text{ad}} - \left( \frac{dT}{dz} \right) \right] \tag{1}
\]

now describes the time-dependent temperature contrast b/n parcel and surroundings as the parcel moves.

\[
\text{(B) The Eqn. of Motion & viscous drag.}
\]
To derive \( \dot{V} \) in above leakage equation: \( \frac{dv}{dt} = \frac{P - P'}{P} g \) (buoyancy only)

The Rayleigh-Bénard case of incompressible convection, the Eqn. of Motion:

\[
\left( \frac{\partial}{\partial t} + v \cdot \nabla \right) v = -\frac{1}{\rho} \nabla p - g + \nu \nabla^2 v
\]

To ensure circulatory motion: \( \nabla \cdot \mathbf{v} = 0 \)

\( \nu = \) kinematic viscosity (same units as \( \nu_k \) [cm/s])

also replace Laplacian \( \nabla^2 v \approx \nu / \ell^2 \).

\[ \Rightarrow \quad \frac{dv}{dt} = \frac{P - P'}{P} g - \nu \frac{v}{\ell^2} \]

\( \uparrow \) always decelerates the parcel

From \( \rho \) to \( T \):

\[ -\frac{\rho - \rho'}{\rho} = -Q \frac{T - T'}{T} = \frac{Q}{T} \Delta T \]

\[ Q = \frac{x_k}{x_k'} \quad (P = P_0 x_k T) \]

Eqn. Motion:

\[ \frac{dv}{dt} = \frac{Qk}{T} \Delta T - \frac{\nu v}{\ell^2} \]  \( (2) \)

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Convective Efficiency

Neutral Stability against convection applies when both

\[ \frac{D\Delta T}{Dt} = 0 \quad \text{and} \quad \frac{dv}{dt} = 0 \]
Ra = \frac{\alpha g L^4}{\nu (\frac{dT}{dz})_{ad} - (\frac{dT}{dz})} = 1

This dimensionless quantity is the Rayleigh number, showing how well the driving (e.g., \( g \frac{dT}{dz} \)) compares to damping (\( \nu \) and \( \nu_f \)).

More on Ra: If a blob is buoyant, it is because of density contrast \( \rho_b \) will feel a force \( \rho_b g \), and its rise time will be \( t_{\text{buoy}} = (\frac{\rho_b g}{\rho_b g})^{\frac{1}{2}} \).

\[ Ra \propto \frac{t_{\text{visc}}}{t_{\text{buoy}}} \]

Ratio of the buoyancy to the geometric mean "heat-viscous" time scale is proportional to \( (Ra)^{-\frac{1}{2}} \).

or \[ Ra \propto \frac{\text{buoyancy force}}{\text{viscous force}} \]

For \( Ra > 1 \), both \( \frac{\Delta T}{T} \) and \( \frac{dv}{t} \) have exponentially growing solutions, i.e., a parcel is accelerated and becomes even hotter than its surroundings \( \Rightarrow \) any perturbation is amplified.

In labs, convection sets in for \( Ra \geq 10^3 \). This departure of \( 10^3 \) is due to the nonlinearity ignored by the Boussinesq approx. & the 6th order diff.

equation derived (CIT Phys 135/47.38) \( \Rightarrow \) air in a room \( Ra = 3 \times 10^8 (\Delta T/1K)(\ell/1m)^3 \).

\[ \text{Def: Prandtl number: } Pr = \frac{\nu}{\alpha \nu} \]

measures the relative importance of viscous damping effects to those of thermal diffusion — property of material, not flow.
Def: *Brunt-Väisälä frequency* — the natural frequency of a medium with a density gradient (in one dimension):

\[ N^2 = -\frac{g}{\rho} \frac{\partial \rho}{\partial z} \]

Also called *buoyancy frequency*.

The mechanical response of an atmosphere is given by this frequency.
\[ \Lambda = Ra \times Pr = \frac{Q q l^4}{T x_t} \left[ \frac{dT}{dz} \right]_{ad} - \left[ \frac{dT}{dz} \right] \]

- convective efficiency.

\[ \sigma - \text{complex angular frequency.} \]

Assume that all coefficients in Equations Energy & Motion are constant
\[ \Rightarrow \text{solutions of form } \Delta T \text{ or } V \propto e^{\sigma t} \]

Combine eqns (1) and (2) into a single second-order eqn, substitute \( e^{\sigma t} \), and find that \( \sigma \) must satisfy:

\[ \sigma^2 + \sigma \frac{y_t}{\ell^2} (1 + Pr) + \frac{y_t y}{\ell^4} (1 - Ra) = 0 \]

for \( Ra < 0 \) no convection, because

\[ \Rightarrow \text{real part of } \sigma < 0 \]

⇒ \( V \) and \( \Delta T \) decrease exp., convection dies out

For \( Ra > 1 \) always one positive real root.

Neglect \( Pr \), and discard the 1 next to \( Ra \),

\[ \sigma^2 + \frac{y_t}{\ell^2} = \frac{Q}{T} \left[ \frac{d^2}{dz^2} - \left( \frac{d^2}{dz} \right)^7 \right] = 0 \]

\[ \text{Def. Brunt–Väisälä frequency} \]

\[ \nu_2 \]
\[
\left(\frac{\sigma}{-N}\right)^2 + \frac{1}{\sqrt{\Lambda}} \left(\frac{\sigma}{-N}\right) - 1 = 0
\]

roots:
\[
\frac{\sigma}{-N} = -\frac{1}{2\sqrt{\Lambda}} + \frac{1}{2} \left(\frac{1}{\Lambda} + 4\right)^{1/2}
\]

\(\sigma\) is real & positive


Remember the "stability diagram" from last time:
It is clear from the above equation that $N^2 < 0$ would imply convective instability. Let use for convenience:

\[ N^2 = -\Sigma^2 \]

Now, the convective efficiency is:

\[ \Lambda = -\frac{N^2 \ell^4}{\nu^2} = \frac{\Sigma^2 \ell^4}{\nu^2} \quad \text{and for } \Sigma: \]

\[ \sigma^2 + \frac{\nu}{\ell^2} \sigma - \Sigma^2 = 0 \]

or

\[ \left( \frac{\sigma}{\Sigma} \right)^2 + \frac{1}{\Lambda \ell^2} \left( \frac{\sigma}{\Sigma} \right) - 1 = 0 \]

where we can derive roots

\[ \Rightarrow \frac{\sigma}{\Sigma} = -\frac{1}{2 \Lambda \ell^2} + \frac{1}{2} \left( \frac{1}{\Lambda} + 4 \right)^{1/2} \]

with the signs chosen, $\Lambda > 1$ (conv. inst.) the $\Sigma$ is real & "+".

Notice that $\Lambda \sim (|N| \times \text{cooling time})^2$ since $\frac{\ell^4}{\nu^2} = \frac{t^2}{\text{cool}}$

Thus $\Lambda \gg 1$ implies that $t_{\text{cool}}$ is long compared to $t_{\text{conv}}$

where $t_{\text{conv}} \sim \frac{1}{|N|}$, hence convection is efficient.

We have 3 cases:

1. $\Lambda \gg 1$ means that $t_{\text{cool}} \gg t_{\text{conv}}$ efficient
   hence $\frac{\sigma}{\Sigma} \approx 1$ from which we have:

\[ \Sigma^2 = \ell^2 (|N| \times \text{cooling time})^2 \]

(a) If $N^2 < 0$, then $\Sigma = |N|$ and $\sigma$ are both real,

\[ \Rightarrow \nu \text{ and } \Delta T \text{ grow exponentially with a time scale } \frac{1}{|N|} \]

(b) If $N^2 > 0$, then $\Sigma$ and $\sigma$ are imaginary,

\[ \Rightarrow \nu \text{ and } \Delta T \text{ oscillate with freq. } N \text{ and with no growth or decay of the motion.} \]

In other words, $t_{\text{cool}} \to \infty$ (adiab.) - gravity waves.
\[ \frac{d\Delta T}{dt} = \Delta T = \sqrt{\left(\frac{dT'}{dz}\right)_{ad} - \left(\frac{dT}{dz}\right)} - \frac{v_t}{e^2} \Delta T \]

\[ \Delta T = \frac{\sigma e \left[ I \right]_{ad} - (\cdot) \left[ I \right]}{\sigma + \frac{v_t}{e^2}} = \frac{\sigma^2 e \left[ I \right]}{N^2} = \frac{\sigma^2 e T}{Q_g} \]
\[ V = -N \Lambda^{1/2} \lambda \]

2. \( \Lambda \ll 1 \) inefficient convection \( \Rightarrow L \approx N^{1/2} \)
   (a) convective case \( \Rightarrow N \Lambda \gg 1 \)
   (b) nonconvective case \( \Rightarrow N \Lambda \ll 1 \)

3. Intermediate case \( \Rightarrow N \Lambda \approx 1 \)
   parcel level \( \Rightarrow \) damped heat by radiation
   Semi-periodic behavior

(D) CONVECTIVE FLUXES & CALCULATIONS

With MLT.

The parcel survives a distance \( \lambda \), when it merges with the surroundings and releases its residual heat.

The heat released \( \Rightarrow \rho c_p \Delta T \) [erg cm\(^{-3}\)]

\( c_p \) because pressure equil.

Rate of heat release will be:

\[ V \rho c_p \Delta T \] [erg cm\(^{-2}\) s\(^{-1}\)]

which is \( F_{\text{conv}} \).

\[ F_{\text{conv}} = \rho v c_p \Delta T = \frac{\rho c_p TR^3 \tau^2}{Q_g} \]

For efficient convection:

\( N \approx 1, \sigma \approx N \)

\[ F_{\text{conv}} = \frac{\rho c_p T R^3 \tau^2}{Q_g} = \frac{\rho c_p T R^3 \tau^2}{Q_g} \frac{\lambda_p^{3/2} \left[ \frac{\partial \ln T}{\partial \ln P} - \left( \frac{\partial \ln T'}{\partial \ln P} \right)_{ad} \right]^{3/2}}{\lambda_p^{3/2}} \]

(E) MLT Approximations:

1. The parcel has the same size as the mixing length \( \lambda \).
2. The mixing length is much shorter than any scale length associated with the stellar structure, e.g. \( \lambda_p \).
3. \( P = P' \), i.e. \( \tau \ll t_{\text{survival}} \).
4. Acoustic phenomena ignored; also shocks.
5. Small \( \delta T \) & \( \delta P \) between parcel & medium.
(F) Applications -

Most often the mixing length $\ell$ is given in terms of the pressure scale height:

$$\ell = \alpha \Delta p$$

and the mixing length parameter $\alpha$, is specified (of order unity).

1. Deep Convection in Massive Stars

Local convective instability:

$$\nabla_{\text{rad}} > \nabla_{\text{ad}} \quad \frac{\frac{d \ln T}{d \ln P}}{\frac{d \ln T}{d \ln P}}_{\text{rad}} = \frac{3 \frac{P_k}{16\pi ac G} \frac{L_{\text{rad}}}{T^4}}{M_r}$$

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CNO: \frac{d E_{\text{CNO}}}{d \ln \rho} = \frac{3L_r}{\Omega} \quad \text{get large } \nabla_{\text{rad}}, \quad \kappa = 0.34 \text{ cm}^{-2} \text{ s}^{-1}
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ZAMS $M \approx 1.5 M_\odot$ for $M = 30 M_\odot$, conv. carries $90\%$

Core conv. is efficient (adiabatic due to large cooling time, $10^7$ yrs, $\approx \frac{M}{\dot{M}}$)

but subsonic - $V \approx 5 \times 10^4 \text{ cm/s}$, $V_\text{s} = 10^6 \text{ cm/s}$

2. Envelope Convection Zones

RGB & AGB - eco value of $\nabla_{\text{rad}}$ in H ionization zone -

- density and $T$ gradients are shallow
- $\nabla \approx 100 \text{ cm}^2 \text{ s}^{-1}$
- $\nabla_{\text{rad}} = 1000$, $\nabla_{\text{ad}} = 0.1$, $\nabla \times 0.3$

But $V \approx 2.5 \times 10^5 \text{ cm s}^{-1}$ while $V_\text{s} \approx 5 \times 10^5 \text{ cm/s}$ $T_{\text{conv}} \times 1 \text{ yr}$

3. Overshoot - parcel with $V_0$ at $r_0$ decelerates due to neg. buoy.

So at $r > r_0$:

$$V^2(r) = V_0^2 + 2 \int_{r_0}^r \frac{g(r) \Delta T}{\Delta r} \, dr$$

R. Denneyee - rotation & 2D

4. Semi-convection - conv. core moves outwards with evolution of radial

mix material and form an abundance (or $\mu$) step

$\rightarrow$ discontinuity in $\nabla_{\text{rad}} / \nabla_{\text{ad}}$
**BEYOND MLT**

Convection is, a priori, nonlocal & non-linear.

- MLT is essentially a compressible treatment of the turbulent transport of heat, a local theory, closer in spirit to Kolmogorov theory of turbulence — $V_k \propto l_{k}^{1/3}$ — hierarchy of eddies, for which $V_k$ & $l_k$ scale.

- The CM model of stellar turbulent convection
  
  The turbulence spectral function $E(k)$ is determined from buoyancy.

Show CM plots.

- Complete 3-D simulations — envelope zones

  - Topology is different — warm broad upflows with cool narrow downdrafts.

  - Horizontal mass flux divergence $\frac{\partial \rho u_x}{\partial x} + \frac{\partial \rho u_y}{\partial y}$ must balance the vertical one $\frac{\partial \rho u_z}{\partial z}$

  The horizontal flow is converging in descending gas and diverging in ascending gas.

1. $\Rightarrow$ Introduces a fundamental asymmetry between up and down — narrow downdrafts.

2. The entropy of the ascending gas increases with height; of descending gas with depth (due to entrainment) — see later.

   $\Rightarrow$ Net $S$ increases through lower conv. zone, but downdrafts are cool — hence buoyancy.

Work is done (Fkm) Fig. 8
Stellar Convection: Issues

The main uncertainties in stellar structure due to the influence of convection may be conveniently discussed with reference to Fig. 1. The distorted radius scale brings out the rapid variation of the convective instability with depth in the surface layers. It is the structure of the temperature gradient in a narrow surface region (a few tens of kilometers) that almost entirely determines the entropy in the bulk of the convection zone.

Figure 1: Logarithmic temperature gradients in a standard model of the solar convection zone (courtesy J. Christensen-Dalsgaard). The dashed line is the adiabatic temperature gradient, the dotted line is the radiative temperature gradient, and the full drawn one is the temperature gradient adopted in a local mixing length model.
manner. From the astrophysical point of view, it requires a

FIG. 5.—Dimensionless temperature gradient, $d \log T/d \log P$, vs. pressure in the upper convective layer of the Sun. The solid line corresponds to the present model. The dotted line corresponds to the CM model. The MLT (with $\alpha = 1.55$) yields quite different results represented by the dashed line.

FIG. 6.—Isochrones in the HR diagram computed with the present model for an extreme Population II chemical composition ($Y = 0.23$ and $Z = 10^{-4}$). Squares mark the fiducial turn-off region for the very metal-poor globular cluster M68.
Figure 8: Convective, radiative, kinetic energy and viscous fluxes plus the net total flux and the entropy per unit mass. The entropy is nearly constant throughout the convection zone, but has a slightly stable gradient in the lower part of the zone, where the radiative flux is gradually decreasing and heating the gas as the Kramer’s opacity increases with decreasing temperature. Near the top of the convection zone most of the flux is carried by the enthalpy or convective flux. Below the convective layer the convective flux becomes negative (transports energy downward) and the radiative flux becomes larger than the total net flux to compensate.

downdrafts) keeps increasing down to the bottom of the convection zone proper. Note the large difference in entropy contrast between the top and bottom of the model. The difference is still much less than in the real Sun, because our upper boundary is at some depth below the real surface.

Fig. 9 thus illustrates that the entropy fluctuations injected into the ascending flow at the bottom of the convection zone are much smaller than the ones injected at the top. Since entropy fluctuations in the ascending gas are rapidly washed out by horizontal expansion, it is clear why the picture in the bulk of the convection zone is one with intermittent patches of cold descending gas, against a smooth background of ascending gas. In this respect, solar convection resembles convection in the Earth’s oceans, with descending thermals because in both cases the entropy fluctuations are produced at the upper boundary. The Earth’s atmosphere, on the other hand, has rising thermals because it is primarily heated from below.

Figure 10 illustrates the overall situation. The vertical plane shown here goes through one of the few (≈ 2–3) penetrating downdrafts that exist at any one time in our model (which is 400 × 400 Mm horizontally). In this particular vertical plane one can see the downdraft penetrating into the stable layer, spreading out a bit, and injecting a return flow back into the convection zone. The entropy of the return flow is mostly similar to that in the convection
Figure 9: A two-dimensional histogram of entropy. Each pixel column is the probability distribution function (PDF) of entropy at given radius.

Figure 10: The entropy in a 400 × 400 × 350 Mm "toy-model" of the solar convection zone (only about 220 Mm is shown vertically). The model is Cartesian instead of spherical, and the flux is about 10⁶ times solar, but the radial profile of radiative diffusion is roughly proportional to the solar one (a scaled version of Kramers opacity is being used), and the vertical density and pressure stratification are similar to solar. The overall extent of the model, including boundary layers and stable layers not shown here, is about 14 pressure scale heights.